Rate functions for Markov chains on finite spaces

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1 Introduction

Consider a finite set $S = \{1, 2, \dots, N\}$, and the associated space of probability measures $\mathcal{M}_1(S)$. Then consider a simple Markov chain $(X_i)_{i\geq 1}$ on S with transition kernel $\Pi_{ij} = \mathbb{P}\{X_{k+1} = j | X_k = i\}$ and initial measure $\mu_0 \in \mathcal{M}_1(S)$. Let $L_n^X = (L_n^X(1), \dots, L_n^X(N))$ denote the empirical measure defined by

$$L_n^X(i) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{X_k = i\}}. \tag{1.1}$$

It has values in $\mathcal{M}_1(S)$. One interest of this sequence is that it contains useful information to study the asymptotic behavior of the Markov chain. For instance, the empirical mean of $f(X_n)$ where f

is any function $S \to \mathbb{R}$ is simply $\int f(x) L_n^X(\mathrm{d}x)$. We try to understand the asymptotic behavior of the distribution of L_n^X in $\mathcal{M}_1(S)$.

A sequence of distributions (p_n) over a topological space is said to satisfy a large deviation principle (LDP) with rate function I if for every Borel set B,

$$-\inf_{x \in B^\circ} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log p_n(B) \leq \limsup_{n \to \infty} \frac{1}{n} \log p_n(B) \leq -\inf_{x \in \overline{B}} I(x). \tag{1.2}$$

The value $I(x)$ should be understood as the rate of exponential decay of the probability to be close to x, under law p_n. The closer it gets to 0, the higher is the probability for a sequence of random variable of laws (p_n) to get near x. In the case of L_n^X, the formulation of the LDP with rate function I is, for every B Borel set of $\mathcal{M}_1(S)$,

$$-\inf_{q \in B^\circ} I(q) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_n^X \in B) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_n^X \in B) \leq -\inf_{q \in \overline{B}} I(q). \tag{1.3}$$

In the following, we state and try to understand the LDP for L_n^X over $\mathcal{M}_1(S)$ when Π is irreducible. This has been widely discussed in the literature, and the following document only applies existing results to the irreducible Markov chains over finite state space, possibly refining and detailing them in this specific case. Most of the statements are adapted from [DZ10] and [RAS15].

In this document, we will prove the LDP for empirical measures as an application of the Gärtner-Ellis Theorem as done in [DZ10]. But there are other ways to prove it (from [RAS15] or [dH08] for instance), each one providing a different rate function. It can even happen that along the proof, the upper bound of (1.2) is obtained with a certain rate function, and the lower bound is obtained with another. In order to prove a proper LDP, one must show that these two functions are equal! We will study four functions that typically appear in LDP proofs, and show that they are equal. Our goal is to understand how comes that these functions are equal, and understand precisely the relations between them.

The four functions are all expressed under a variational form, and we will try to manipulate their maximizers and minimizers when they exist in order to detail their behavior and relations between them. These descriptions help to understand what happens when they do not exist.

A deep comprehension of the relations between the four studied rate functions in simple cases should also help us deal with the critical cases to come (when Π is no longer irreducible for instance).

The assumption that Π is irreducible is of great help to keep the Markov chain as simple as possible. In the following, we mainly work under the irreducibility assumption:

(Irr) The matrix Π is irreducible, *i.e.*,

$$\forall i, j \in S, \exists p \in \mathbb{N} \quad \Pi^p(i,j) > 0.$$

For many statements, the following positivity assumption will be crucial:

(Pos) All the entries of Π are positive.

Of course, this is a stronger assumption on Π, and we shall prefer **(Irr)** to it whenever possible.

2 LDP for empirical measures

For $\lambda \in \mathbb{R}^S$, let Π_λ and $\tilde{\Pi}_\lambda$ be the tilted matrices defined from Π by $\Pi_\lambda(i,j) = e^{\lambda_i}\Pi_{ij}$ and $\tilde{\Pi}_\lambda(i,j) = e^{\lambda_j}\Pi_{ij}$. For the needs of Theorem 2.7, we are interested in the value of their spectral radius denoted $\rho(\Pi_\lambda)$ and $\rho(\tilde{\Pi}_\lambda)$.

2.1 Preliminary remarks on Π_λ, $\tilde{\Pi}_\lambda$, and the Perron-Frobenius theorem

We should first understand that Π_λ, $\tilde{\Pi}_\lambda$ are essentially equivalent from a spectral point of view.

Proposition 2.1. *Let $D = \mathrm{diag}(e^{\lambda_1}, \dots, e^{\lambda_N})$. Then, $u \mapsto Du$ is a bijection between the set of eigenvectors of $\tilde{\Pi}_\lambda$ and the set of eigenvectors of Π_λ that preserve the associated eigenvalue. Moreover,*

$$\rho(\Pi_\lambda) = \rho(\tilde{\Pi}_\lambda) = \rho(\Pi_\lambda^T) = \rho(\tilde{\Pi}_\lambda^T). \tag{2.1}$$

Proof. Notice that one can rewrite Π_λ as $\Pi_\lambda = D\Pi$ and $\tilde{\Pi}_\lambda$ as $\tilde{\Pi}_\lambda = \Pi D$. If u is an eigenvector of $\tilde{\Pi}_\lambda$ associated to the eigenvalue α, the equality $\Pi D u = \alpha u$ yields $D\Pi(Du) = \alpha(Du)$, meaning that Du is an eigenvector of $D\Pi = \Pi_\lambda$ associated to the eigenvalue α. Now if v is an eigenvector of Π_λ associated to the eigenvalue α, note that $D^{-1}v$ is an eigenvector of $\tilde{\Pi}_\lambda$ associated to the eigenvalue α, because $\Pi D(D^{-1}v) = D^{-1}(D\Pi)v = D^{-1}\alpha v$. As Π_λ, $\tilde{\Pi}_\lambda$, and their transpose have the same eigenvalues, they have the same spectral radius. □

Let us recall the Perron-Frobenius theorem as stated in [DZ10, Theorem 3.1.1]. It can help to compute the spectral radius of irreducible non-negative matrices.

Theorem 2.2 (Perron-Frobenius)**.** *Let A be an irreducible non-negative matrix indexed in $S \times S$. Then $\rho(A)$ is a simple eigenvalue (called the Perron-Frobenius eigenvalue) of A, such that*

1. *A has left and right eigenvectors (called Perron-Frobenius eigenvectors) associated to the eigenvalue $\rho(A)$, that have positive coordinates,*
2. *the left and right Perron-Frobenius eigenvectors are unique up to scalar multiplication,*
3. *for every $i \in S$, for every vector ϕ having all of its coordinates positive,*

$$\log(\rho(A)) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{j\in S} \phi_j A^n(j,i)\right) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{j\in S} A^n(i,j)\phi_j\right). \tag{2.2}$$

Proof. The first two points are well known and are discussed in [HJ93, Section 8.4]. For the last point, let u be the left Perron-Frobenius eigenvector of A, and let $\alpha = \sup_i u_i > 0$, $\beta = \inf_i u_i > 0$, $\gamma = \sup_i \phi_i > 0$, $\delta = \inf_i \phi_i > 0$. Then, for every i, j,

$$\frac{\delta}{\alpha} u_i A^n(i,j) \leq \delta A^n(i,j) \leq \phi_i A^n(i,j) \leq \gamma A^n(i,j) \leq \frac{\gamma}{\beta} u_i A^n(i,j).$$

Taking the sum over j of the above inequalities yields

$$\frac{\delta}{\alpha}\rho(A)^n u_j \leq \sum_{j=1}^{N} A^n(i,j)\phi_i \leq \frac{\gamma}{\beta}\rho(A)^n u_j.$$

Take the logarithm and get

$$\frac{1}{n}\log\left(\frac{\delta}{\alpha}u_j\right)+\log\rho(A)\leq\frac{1}{n}\log\left(\sum_{i=1}^{N}\phi_i A^n(i,j)\right)\leq\frac{1}{n}\log\left(\frac{\gamma}{\beta}u_j\right)+\log\rho(A).$$

Thus taking the limit provides the first equality in (2.2). One can repeat this reasoning with the right Perron-Frobenius eigenvector v of Π_λ to get the second one. □

The particular cases of $A=\Pi_\lambda$ or $A=\tilde{\Pi}_\lambda$ is interesting to note. The following statement holds because Π_λ and $\tilde{\Pi}_\lambda$ are irreducible if and only if Π is.

Corollary 2.3. *Under* **(Irr)***, for a deterministic vector ϕ having all of its coordinates positive, one has*

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i=1}^{N}\phi_i\Pi_\lambda^n(i,j)\right)=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{j=1}^{N}\Pi_\lambda^n(i,j)\phi_j\right)=\log(\rho(\Pi_\lambda)). \tag{2.3}$$

Corollary 2.3 has useful consequences: now $\log\rho(\Pi_\lambda)$ can be expressed as a limit depending on the law of L_n^X. In the following, if q is a measure on S and λ is a vector of $\mathbb{R}^S$, $\langle q,\lambda\rangle$ denotes their product in the duality $\mathbb{R}^S\leftrightarrow\mathbb{R}^S$:

$$\langle q,\lambda\rangle:=\int_S\lambda_i\mathrm{d}q(i)=\sum_{i\in S}q_i\lambda_i. \tag{2.4}$$

Proposition 2.4. *Under* **(Irr)***,*

$$\log\rho(\Pi_\lambda)=\log\rho(\tilde{\Pi}_\lambda)=\log\rho(\Pi_\lambda^T)=\log\rho(\tilde{\Pi}_\lambda^T)=\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[e^{n\langle L_n^X,\lambda\rangle}\right]. \tag{2.5}$$

Proof. The only equality that is not already stated in (2.1) is the last one. Let us compute precisely the expectation in (2.5). Remember the initial state is distributed according to μ_0. We get

$$\begin{aligned}
\mathbb{E}\left[e^{n\langle L_n^X,\lambda\rangle}\right]&=\mathbb{E}\left[e^{\sum_{i=1}^n\lambda_{X_i}}\right]\\
&=\sum_{x_1,\ldots,x_n}\mathbb{P}(X_1=x_1,\ldots,X_n=x_n)\prod_{i=1}^{n}e^{\lambda_{x_i}}\\
&=\sum_{x_0,x_1,\ldots,x_n}\mu_0(x_0)\left(\Pi(x_0,x_1)e^{\lambda_{x_1}}\right)\cdots\left(\Pi(x_{n-1},x_n)e^{\lambda_{x_n}}\right)\\
&=\sum_{x_0=1}^{N}\mu_0(x_0)\sum_{x_1,\ldots,x_n}\tilde{\Pi}_\lambda(x_0,x_1)\cdots\tilde{\Pi}_\lambda(x_{n-1},x_n)\\
&=\sum_{x_0=1}^{N}\sum_{x_n=1}^{N}\mu_0(x_0)\tilde{\Pi}_\lambda^n(x_0,x_n).
\end{aligned}$$

Thus by Corollary 2.3, for any j,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[e^{\sum_{i=1}^n\lambda_{X_i}}\right]=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i=1}^{N}\mu_0(i)\tilde{\Pi}_\lambda^n(i,j)\right)=\log\rho(\tilde{\Pi}_\lambda),$$

and (2.5) is finally proven. □

Applying Hölder inequality to the functions $\lambda \mapsto \frac{1}{n} \log \mathbb{E}\left[e^{n\langle L_n^X, \lambda\rangle}\right]$ yields that each one is convex. Thus, Proposition 2.4 implies that $\lambda \mapsto \log \rho(\Pi_\lambda)$ is convex as a pointwise limit of convex functions. However, a direct way of proving the convexity provides the following stronger statement.

Lemma 2.5. *The function $\lambda \mapsto \log \rho(\Pi_\lambda)$ is strictly convex, that is to say for all λ, λ' and all $0 < t < 1$,*

$$\log \rho\left(\Pi_{t\lambda+(1-t)\lambda'}\right) \le t \log \rho\left(\Pi_\lambda\right) + (1-t) \log \rho\left(\Pi_{\lambda'}\right),$$

and the inequality is strict unless $\lambda - \lambda'$ is constant.

In the previous lemma and in the following, saying that a vector is constant is, by definition, saying that all its coordinates are equal.

Proof. The convexity comes from Hölder inequality. Let $\lambda, \lambda' \in \mathbb{R}^S$ and $0 \le t \le 1$, and let $\Gamma = \Pi_{t\lambda+(1-t)\lambda'}$. Then $\Gamma = \Pi_\lambda^t \Pi_{\lambda'}^{1-t}$. Let v and w be the right Perron-Frobenius of Π_λ and $\Pi_{\lambda'}$ respectively, and let $u_i = v_i^t w_i^{1-t} > 0$. By Hölder inequality,

$$\begin{aligned} \frac{1}{u_i} \sum_j \Gamma(i,j) u_j &= \frac{1}{u_i} \sum_j (\Pi_\lambda(i,j) v_j)^t (\Pi_{\lambda'}(i,j) w_j)^{1-t} \\ &\le \frac{1}{u_i} \left(\sum_j \Pi_\lambda(i,j) v_j \right)^t \left(\sum_j \Pi_{\lambda'}(i,j) w_j \right)^{1-t} \\ &= \frac{1}{u_i} \left(\rho(\Pi_\lambda) v_i\right)^t \left(\rho(\Pi_{\lambda'}) w_i\right)^{1-t} \\ &= \rho(\Pi_\lambda)^t \rho(\Pi_{\lambda'})^{1-t}. \end{aligned} \tag{2.6}$$

Thus by [HJ93, Theorem 8.1.26] (bounds for the spectral radius of a matrix),

$$\rho(\Gamma) \le \max_i \frac{1}{u_i} \sum_j \Gamma(i,j) u_j \le \rho(\Pi_\lambda)^t \rho(\Pi_{\lambda'})^{1-t}. \tag{2.7}$$

This shows the convexity. Now turn to the necessary conditions for inequality (2.7) to be an equality. If it is, there is equality in Hölder inequality (2.6). For all i, equality case in Hölder inequality implies the existence of $\alpha_i \in (0, \infty)$ such that

$$\forall j \in S \quad \Pi_\lambda(i,j) v_j = \alpha_i \Pi_{\lambda'}(i,j) w_j.$$

Summing these equalities over j yields $\rho(\Pi_\lambda) v_i = \alpha_i \rho(\Pi_{\lambda'}) w_i$ so $\alpha_i = \frac{\rho(\Pi_\lambda)}{\rho(\Pi_{\lambda'})} \cdot \frac{v_i}{w_i}$. Therefore,

$$\forall i, j \in S \quad \frac{v_j}{w_j} = \alpha_i e^{\lambda_i - \lambda'_i} = \frac{\rho(\Pi_\lambda)}{\rho(\Pi_{\lambda'})} \frac{v_i}{w_i} e^{\lambda_i - \lambda'_i}.$$

Thus the ratio $\frac{v_j}{w_j}$ is independent of j, and values a certain constant c. The previous equation becomes

$$\forall i \in S \quad c = \frac{\rho(\Pi_\lambda)}{\rho(\Pi_{\lambda'})} c e^{\lambda_i - \lambda'_i}.$$

So $\lambda - \lambda'$ has to be constant. □

2.2 The large deviations principle as application of Gärtner-Ellis Theorem

First we recall the Gärtner-Ellis Theorem in finite dimension.

Theorem 2.6 (Gärtner-Ellis)**.** *Let μ_n be a sequence of laws over $\mathbb{R}^S$, associated with their logarithmic moment generating function Λ_n defined by*

$$\Lambda_n(\lambda) := \log \int e^{\lambda x} \mu_n(\mathrm{d}x), \quad \lambda \in \mathbb{R}^S.$$

Assume for each $\lambda \in \mathbb{R}$ the existence of a pressure $\Lambda(\lambda)$ given by

$$\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{n} \log \int e^{n\lambda x} \mu_n(\mathrm{d}x) \in \mathbb{R} \cup \{\pm\infty\},$$

and assume Λ is steep, lower semicontinuous, and differentiable over its domain. Furthermore, assume that 0 is in the interior of the domain of Λ. Then, μ_n satisfies a LDP with rate function the Legendre-Fenchel transform of Λ.

The Sanov Theorem for the empirical measures of a Markov chain derives form this theorem.

Theorem 2.7 (Sanov)**.** *The distribution of L_n^X satisfies a LDP with rate function the Legendre-Fenchel transform of $\lambda \mapsto \log \rho(\Pi_\lambda)$ over $\mathbb{R}^S$.*

Proof. The existence of the pressure derives from equation (2.5). Plus, it yields an expression for Λ, that is $\Lambda(\lambda) = \log \rho(\Pi_\lambda)$. This function is defined over $\mathbb{R}^S$, lower semicontinuous, and convex by Lemma 2.5. It remains to see that it is differentiable over $\mathbb{R}$. Thanks to [Ser10, Theorem 5.3], the Perron-Frobenius eigenvalue of a matrix, being algebraically simple, is an analytic function of the matrix. Thus Λ is differentiable over $\mathbb{R}$. □

3 Different expressions of the rate function

For any probability vector $q \in \mathcal{M}_1(S)$, consider the following rate functions:

$$I(q) = \sup_{\lambda \in \mathbb{R}^S} (\langle q, \lambda \rangle - \log \rho(\Pi_\lambda)), \tag{3.1}$$

$$J(q) = \sup_{u>0} \sum_i q_i \log \frac{u_i}{(u\Pi)_i}, \tag{3.2}$$

$$K(q) = \sup_{v>0} \sum_i q_i \log \frac{v_i}{(\Pi v)_i}, \tag{3.3}$$

$$L(q) = \inf_{\mathcal{K}_S(q)} \sum_{i,j} q_i Q_{ij} \log \frac{Q_{ij}}{\Pi_{ij}}, \tag{3.4}$$

where $u > 0$, $v > 0$ means that u and v must have their coordinates positive, and where $\mathcal{K}_S(q)$ is the set of stochastic kernels $Q = (Q_{ij})_{i,j \in S}$ over S which admit q as an invariant measure. We name f_I^q, f_J^q, f_K^q, f_L^q the functions being optimized in $I(q), J(q), K(q), L(q)$ respectively.

Let S_q be the support of q, that is to say the set of indices i such that $q_i > 0$.

Remark 3.1. Let us check the above definitions. Under **(Pos)** assumption, there is no difficulty in the definitions of f_I^q, f_J^q, f_K^q, and f_L^q. However, under **(Irr)**, we should check definitions and set conventions. The coordinates of Πv are positive for positive vectors v (otherwise there exists i such that the whole i-th line of Π is null, which is impossible because the lines of Π sum up to 1), so $f_K^q(v)$ is well defined and finite. The coordinates of $u\Pi$ are positive for positive vectors u (otherwise there exists a j such that the whole j-th column column of Π is null, which implies that the j-th column of Π^p is null for every p, and it is impossible because of **(Irr)**), so $f_J^q(u)$ is well defined and finite. Notice that the conclusion $u\Pi > 0$ used **(Irr)** but $\Pi v > 0$ is only based on the fact that Π is stochastic. In the reducible case, we work with the convention $q_j \log \frac{u_j}{0} = +\infty$ if $q_j > 0$ and $q_j \log \frac{u_j}{0} = +\infty = 0$ if $q_j = 0$.

As for f_L^q, we will work under the convention $0 \log 0 = 0$. This disambiguates the definition of $f_L^q(Q)$ for a stochastic kernel Q that is absolutely continuous with respect to Π (that is, if $\Pi(i,j) = 0$ for some $i, j \in S$, then $Q(i,j) = 0$). When there exsit $i, j \in S$ such that $\Pi(i,j) = 0$ and $Q(i,j) \neq 0$, we take $\frac{Q(i,j)}{\Pi(i,j)}$ to be $+\infty$. If i is such that $q_i > 0$, it yields $f_L^q(Q) = +\infty$. The function f_L^q is only finite over the set of stochastic kernels that satisfy $\forall i \in S_q, \forall j \in S \quad \Pi(i,j) = 0 \Rightarrow Q(i,j) = 0$. As long as this set is not empty, $L(q) < +\infty$.

In the previous section, we showed the LDP for the empirical measures L_n^X associated with the rate function I under **(Irr)**. However, there are other ways to show it. Another proof of the same LDP in [dH08, Theorem 4.6] provides a LDP associated with the rate function L. The proof in [RAS15, Theorem 13.5] ends up with a rate function K for the upper bound and L for the lower bound. In this context, if K was not proven equal to L, the LDP for L_n^X would not be granted. In this section, we will show that, even without **(Irr)**, $I = J = K = L$.

3.1 Equality of I and K

Proposition 3.2. *For all $q \in \mathcal{M}_1(S)$, $I(q) = K(q)$.*

Proof. Let $q \in \mathcal{M}_1(S)$ we show that $I(q) = K(q)$ by showing the two inequalities. In the following, $||\cdot||$ denotes the subordinate matrix norm, associated with the vector norm $|w| = \max_i |w_i|$. It is multiplicative an satisfies for any matrix A and vector w, $|Aw| \leq ||A|| \times |w|$.

$\boxed{I(q) \leq K(q).}$ Let $\lambda \in \mathbb{R}^S$. Take $\alpha > \log \rho(\Pi_\lambda)$, and let

$$v := \sum_{k=0}^{\infty} e^{-k\alpha} \Pi_\lambda^k 1. \tag{3.5}$$

The definition of v comes from [DS89, Lemma 4.1.36]. Now we want to check that v has been properly defined. One has

$$|e^{-k\alpha}\Pi_\lambda^k 1| \leq e^{-k\alpha}||\Pi_\lambda^k|| \; = \exp(-k\alpha + \log(||\Pi_\lambda^k||).$$

The norm $||\cdot||$ being a subordinate norm, one has $\frac{1}{k}\log||\Pi_\lambda^k|| \xrightarrow[k\to\infty]{} \log\rho(\Pi_\lambda)$ (proof in [Ser10, Proposition 7.8]). Thus

$$\exp(-k\alpha + \log(||\Pi_\lambda^k||) = \exp\left(-k\Big(\alpha - \log\rho(\Pi_\lambda) + \underset{k\to\infty}{o}(1)\Big)\right).$$

For k big enough, the factor of k in the exponential is lower than $\beta = \frac{1}{2}(\alpha - \log\rho(\Pi_\lambda)) > 0$. Thus for k big enough, $|e^{-k\alpha}\Pi_\lambda^k 1|$ is dominated by $\left(e^{-\beta}\right)^k$. This justifies that the series $\sum_k e^{-k\alpha}\Pi_\lambda^k 1$ converges.

The matrix Π being stochastic, at least one of the coefficients of each line of Π must be positive, thus for each i, $(\Pi_\lambda 1)_i > 0$, so $v > 0$ (remember Remark 3.1). Then the same arguments yields that $\Pi_\lambda v > 0$ and $\Pi v > 0$. Moreover, v satisfies

$$\Pi_\lambda v = \sum_{k=0}^{\infty} e^{-k\alpha} \Pi_\lambda^{k+1} 1 = e^{\alpha} \sum_{k=1}^{\infty} e^{-k\alpha} \Pi_\lambda^{k} 1 = e^{\alpha}(v-1), \tag{3.6}$$

so we even have $v > 1$. Thus,

$$\begin{aligned} \langle \lambda, q\rangle - f_K^q(v) &= \sum_{i\in S} q_i \log \frac{e^{\lambda_i}(\Pi v)_i}{v_i} \\ &= \sum_{i\in S} q_i \log \frac{(\Pi_\lambda v)_i}{v_i} \\ &= \sum_{i\in S} q_i \left(\log \frac{v_i - 1}{v_i} + \alpha \right) \\ &\le \sum_{i\in S} q_i \alpha = \alpha. \end{aligned} \tag{3.7}$$

This says that $\langle \lambda, q\rangle - \alpha \le f_K^q(v) \le K(q)$, thus by taking the limit when $\alpha \to \log \rho(\Pi_\lambda)$, one has $f_I^q(\lambda) \le K(q)$, satisfied for every λ. Finally, taking the supremum over λ, $I(q) \le K(q)$.

$\boxed{I(q) \ge K(q).}$ Let v be any positive vector. Define λ by $\lambda_i = \log \frac{v_i}{(\Pi v)_i}$. λ is finite because $\Pi v > 0$. We have

$$(\Pi_\lambda v)_i = \sum_j \frac{v_i}{(\Pi v)_i} \Pi(i,j) v_j = v_i,$$

so v is an eigenvector of Π_λ and for all n, $\Pi_\lambda^n v = v$. This implies that $\log \rho(\Pi_\lambda) \le 0$. Indeed, by [Ser10, Proposition 7.8],

$$\log \rho(\Pi_\lambda) = \lim_{n\to\infty} \frac{1}{n} \log ||\Pi_\lambda^n|| = \lim_{n\to\infty} \frac{1}{n} \log \left(\sup_{|w|=1} |\Pi_\lambda^n w| \right).$$

Take a look at $|\Pi_\lambda^n w|$ when $|w| = 1$. It is the maximum over i of $|\sum_j \Pi_\lambda^n(i,j) w_j|$. For each i, the triangle inequality says that this quantity is greater if the coordinates w_i all have the same sign. Thus when optimizing it, one can only consider the w with non-negative coordinates, and

$$\log \rho(\Pi_\lambda) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sup_{|w|=1} |\Pi_\lambda^n w| \right) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sup_{\substack{|w|=1 \\ w\ge 0}} |\Pi_\lambda^n w| \right).$$

Consider some $w \ge$ with $|w| = 1$. As $v > 0$, it satisfies $w \le \frac{1}{\inf_i v_i} v$. Therefore,

$$0 \le \Pi_\lambda^n w \le \frac{1}{\inf_i v_i} \Pi_\lambda^n v = \frac{1}{\inf_i v_i} v,$$

and thus

$$|\Pi_\lambda^n w| \le \frac{|v|}{\inf_i v_i}.$$

This shows that

$$\sup_{\substack{|w|=1 \\ w\ge 0}} |\Pi_\lambda^n w| \le \frac{|v|}{\inf_i v_i}.$$

Taking the logarithm yields $\log \rho(\Pi_\lambda) \leq \lim_{n\to\infty} \frac{1}{n} \log \frac{|v|}{\inf_i v_i} = 0$. Now we have

$$I(q) = \sup_{\lambda \in \mathbb{R}^S} f_I^q(\lambda) \geq f_I^q(\lambda) \geq \langle \lambda, q \rangle - 0 = \sum_i q_i \log \frac{v_i}{(\Pi v)_i} = f_K^q(v). \tag{3.8}$$

Taking the supremum over v provides $I(q) \geq K(q)$. □

3.2 Equality of I and J

The intuition we want to apply to prove that $I = J$ is to try to copy the proof of $I = K$ with left matrix multiplication and $\tilde{\Pi}_\lambda$ instead of Π_λ. But without **(Irr)**, it is possible that $u\Pi$ has null coordinates, even if $u > 0$ (see Remark 3.1), so even if the main arguments are the same, the proof of $I = J$ has to get around this difficulty. Recall we work with the convention $q_j \log \frac{u_j}{0} = +\infty$ if $q_j > 0$ and $q_j \log \frac{u_j}{0} = +\infty = 0$ if $q_j = 0$.

Proposition 3.3. *For all $q \in \mathcal{M}_1(S)$, $I(q) \geq J(q)$.*

Let S' be the set of indices j of non null columns of Π, that is to say

$$S' = \{j \in S \mid \exists i \in S \quad \Pi(i,j) > 0\}. \tag{3.9}$$

In other words, $S \backslash S'$ is the space of states of S which are not reachable from any state under the transition kernel Π. No arrow of the graph of the Markov chain (X_n) points toward them. Observe that if $S' = S$, then the proof of $I = K$ can be copied to show $I = J$.

Lemma 3.4. *If $S' = S$, then for all $q \in \mathcal{M}_1(S)$, $I(q) = J(q)$.*

Proof. Let $q \in \mathcal{M}_1(S)$.

$\boxed{I(q) \leq J(q).}$ Let $\lambda \in \mathbb{R}^S$, and let $\alpha > \log \rho(\Pi_\lambda)$. Define

$$u := \sum_{k=0}^{\infty} e^{-k\alpha} 1\tilde{\Pi}_\lambda^k. \tag{3.10}$$

The series converge with a copy of the argument of Proposition 3.2 because

$$\begin{aligned} |e^{-k\alpha} 1\tilde{\Pi}_\lambda^k| &\leq \exp\left(-k(\alpha - \frac{1}{k}\log ||\tilde{\Pi}_\lambda^k||)\right) \\ &= \exp\left(-k\left(\alpha - \log \rho(\Pi_\lambda) + \underset{k\to\infty}{o}(1)\right)\right). \end{aligned}$$

Now as $S = S'$, every coordinate of $1\tilde{\Pi}_\lambda$ is positive, thus $u > 0$. Therefore, again because of $S = S'$, this implies $u\tilde{\Pi}_\lambda > 0$. Moreover, u satisfies

$$u\tilde{\Pi}_\lambda = \sum_{k=0}^{\infty} e^{-k\alpha} 1\tilde{\Pi}_\lambda^{K+1} = e^\alpha \sum_{k=1}^{\infty} e{-k\alpha} 1\tilde{\Pi}_\lambda^k = e^\alpha(u-1),$$

so $u > 1$. Thus,

$$\begin{aligned} \langle \lambda, q \rangle - f_J^q(u) &= \sum_{j\in S} q_j \left(\log \frac{u_j - 1}{u_j} + \alpha\right) \\ &\leq \sum_{j \in S} q_i \alpha = \alpha. \end{aligned} \tag{3.11}$$

Therefore, $\langle\lambda, q\rangle - \alpha \leq f_J^q(u) \leq J(q)$, and taking $\alpha \to \log\rho(\Pi_\lambda)$ yields $f_I^q(\lambda) \leq J(q)$. Taking the supremum over $\lambda \in \mathbb{R}^S$ yields $I(q) \leq J(q)$.

$\boxed{I(q) \geq J(q).}$ Let $u > 0$. Define λ by $\lambda_j = \log\frac{u_j}{(u\Pi)_j}$. Then λ is finite because $S = S'$. We have

$$(u\tilde{\Pi}_\lambda)_j = \sum_i \frac{u_j}{(u\Pi)_j} u_i \Pi(i,j) = u_j,$$

so u is an eigenvector of $\tilde{\Pi}_\lambda$ and for all n, $u\tilde{\Pi}_\lambda^n = u$. A copy of the argument used in the proof of Proposition 3.2 with the left matrix multiplication and $\tilde{\Pi}_\lambda$ yields that $\log\rho(\Pi_\lambda) = \log\rho(\tilde{\Pi}_\lambda) \leq 0$. Now we have

$$I(q) = \sup_{\lambda\in\mathbb{R}^S} f_I^q(\lambda) \geq f_I^q(\lambda) \geq \langle\lambda, q\rangle - 0 = \sum_{j\in S} q_j \log\frac{u_j}{(u\Pi)_j} = f_J^q(u). \tag{3.12}$$

Taking the supremum over u yields $I(q) \geq J(q)$. □

The assumption that $S' = S$ was crucial in the above argument. Without it, we are not granted that $u > 0$, so the computation of $f_J^q(u)$ is ambiguous and λ has infinite coordinates. Another case that can be easily handeled is when $S_q \not\subset S'$, that is to say there exists an index $j_0 \notin S'$ such that $q_{j_0} > 0$.

Lemma 3.5. *Let $q \in \mathcal{M}_1(S)$. If $S_q \not\subset S'$, then $I(q) = J(q) = +\infty$.*

Proof. In one hand, the j_0-th column of $\tilde{\Pi}_\lambda$ is full of zeros, so $\tilde{\Pi}_\lambda$ is a constant with respect to λ_{j_0}. It implies that the quantity $\rho(\Pi_\lambda) = \rho(\tilde{\Pi}_\lambda)$ do not depends on λ_{j_0}. Thus by taking $\lambda_{j_0} \to +\infty$ and $\lambda_j = 0$ for every other coordinate, one has

$$f_I^q(\lambda) = q_{j_0}\lambda_{j_0} - \log\rho(\Pi_\lambda) \xrightarrow[\lambda_{j_0}\to\infty]{} +\infty.$$

In the other hand, for every $u > 0$,

$$f_J^q(u) = \sum_{j\in S_q} q_j \log\frac{u_j}{(u\Pi)_j} = +\infty,$$

the term of index j_0 being $q_{j_0}\log\frac{u_j}{0} = +\infty$. Thus $J(q) = +\infty$. □

Now we turn to the proof of Proposition 3.3. To prove Proposition 3.3, we use a recurrence argument based on Lemmas 3.4 and 3.5. Let $S^{(0)} = S$, $S^{(1)} = S'$, and define

$$S^{(k+1)} = \{j \in S^{(k)} \mid \exists i \in S^{(k)} \quad \Pi(i,j) > 0\}, \tag{3.13}$$

for $k \geq 1$. In other words, $S\backslash S^{(k)}$ is the set of states that are not the end of any path of n steps in the graph of the Markov chain (X_n).

In the following, we will restrain the state space to $S^{(k)} \subset S$. It means we project $\mathbb{R}^S$ onto $\mathbb{R}^{S^{(k)}}$, and work with extracted matrices[1]. If A is a matrix over $S \times S$, then $A_{S^{(k)}}$ denotes the extracted matrix over $S^{(k)} \times S^{(k)}$ from A. The matrix multiplication is defined by

$$(A_{S^{(k)}v})_i = \sum_{j\in S^{(k)}} A(i,j)v_j, \qquad (uA_{S^{(k)}})_j = \sum_{i\in S^{(k)}} u_i A(i,j). \tag{3.14}$$

[1] we do not re-index the coefficients of $\Pi_{S^{(k)}}$ after extraction.

If $S_q \subset S^{(k)}$, q can be seen as a probability over $S^{(k)}$ and we define slightly modified versions of $I(q)$ and $J(q)$ with

$$I^{(k)}(q) = \sup_{\lambda \in \mathbb{R}^{S^{(k)}}} \left(\sum_{i \in S^{(k)}} q_i \lambda_i - \log \rho \left(\Pi^{\lambda}_{S^{(k)}} \right) \right), \tag{3.15}$$

$$J^{(k)}(q) = \sup_{u \in S^{(k)}} \sum_{i \in S^{(k)}} q_i \log \frac{u_i}{(u \Pi_{S^{(k)}})_i}. \tag{3.16}$$

Let us consider a new Markov chain $\left(X_n^{(k)}\right)$ over $S^{(k)}$, with transition kernel $\Pi_{S^{(k)}}$. We check that it is really a stochastic matrix. As it is an extracted matrix from Π, it is non-negative. Moreover, for all $j \in S^{(k)}$, and for all $i \in S^{(k-1)} \backslash S^{(k)}$, by definition $\Pi_{S^{(k-1)}}(i,j) = 0$. It implies that for all $i \in S^{(k)}$,

$$\sum_{j \in S^{(k)}} \Pi_{S^{(k)}}(i,j) = \sum_{j \in S^{(k-1)}} \Pi_{S^{(k-1)}}(i,j) = \ldots = 1,$$

by a recurrence argument. Thus $\left(X_n^{(k)}\right)$ is a Markov chain defined over $S^{(k)}$. All the previous statements are in force with this new Markov chain. Trying to define the rate functions I and J over $\mathcal{M}_1(S^{(k)})$ for this Markov chain leads naturally to the definitions (3.15) and (3.16).

Proof of Proposition 3.3. Let $q \in \mathcal{M}_1(S)$ and let $k \geq 0$ such that $S_q \subset S^{(k+1)}$. One can see q as an element of $\mathcal{M}_1(S^{(k+1)})$. Start by showing $I^{(k)}(q) = I^{(k+1)}(q)$ and $J^{(k)}(q) = J^{(k+1)}(q)$. This will allow us to trade the problem of showing $I^{(k)}(q) = J^{(k)}(q)$ for the smaller one of $I^{(k+1)}(q) = J^{(k+1)}(q)$.

$\boxed{I^{(k)}(q) = I^{(k+1)}(q).}$ As $S_q \subset S^{(k+1)}$, it is enough to show that $\log \rho(\Pi^{\lambda}_{S^{(k+1)}}) = \log \rho(\Pi^{\lambda}_{S^{(k+1)}})$. To simplify the notations in the following computations, one can assume that, up to reindexation of the sates, $S^{(k)} \backslash S^{(k+1)} = \{1, \ldots, p\}$ and $S^{(k+1)} = \{p+1, \ldots, l\}$. The matrix $\Pi^{\lambda}_{S^{(k)}}$ has the following form:

$$\Pi^{\lambda}_{S^{(k)}} = \begin{pmatrix} (0) & A \\ (0) & \Pi^{\lambda}_{S^{(k+1)}} \end{pmatrix},$$

with a certain matrix A of dimensions $p \times (l-p)$. Thus, for a vector w written by blocs $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, one has

$$(\Pi^{\lambda}_{S^{(k)}})^n w = \begin{pmatrix} (0) & A(\Pi^{\lambda}_{S^{(k+1)}})^{n-1} \\ (0) & (\Pi^{\lambda}_{S^{(k+1)}})^n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} A(\Pi^{\lambda}_{S^{(k+1)}})^{n-1} w_2 \\ (\Pi^{\lambda}_{S^{(k+1)}})^n w_2 \end{pmatrix}.$$

Taking the supremum over $|w| = 1$ yields

$$||(\Pi^{\lambda}_{S^{(k)}})^n|| = \max \left(||A(\Pi^{\lambda}_{S^{(k+1)}})^{n-1}||, ||(\Pi^{\lambda}_{S^{(k+1)}})^n|| \right).$$

Taking the n-th root and the limit when $n \to \infty$ finaly yields $\rho(\Pi^{\lambda}_{S^{(k)}}) = \rho(\Pi^{\lambda}_{S^{(k+1)}})$, thanks to [Ser10, Proposition 7.8]. Therefore, as changing the coordinates λ of indices in $S^{(k)} \backslash S^{(k+1)}$ does not change $\sum_{i \in S^{(k)}} q_i \lambda_i - \log \rho \left(\Pi^{\lambda}_{S^{(k)}} \right)$,

$$\begin{aligned} I^{(k+1)}(q) &= \sup_{\lambda \in \mathbb{R}^{S^{(k+1)}}} \left(\sum_{i \in S_q} q_i \lambda_i - \log \rho(\Pi^{\lambda}_{S^{(k+1)}}) \right) \\ &= \sup_{\lambda \in \mathbb{R}^{S^{(k+1)}}} \left(\sum_{i \in S_q} q_i \lambda_i - \log \rho(\Pi^{\lambda}_{S^{(k)}}) \right) = I^{(k)}(q). \end{aligned} \tag{3.17}$$

$\boxed{J^{(k+1)}(q) = J^{(k)}(q).}$ Let $u > 0$, let $\gamma \in [0,1)$, and take $u' > 0$ defined by

$$u'_j = \begin{cases} u_j & \text{if } j \in S' \\ (1-\gamma)u_j & \text{else.} \end{cases}$$

As $S_q \subset S^{(k+1)} \subset S^{(k)}$, we have

$$\sum_{j\in S^{(k)}} q_j \log \frac{u'_j}{(u'\Pi_{S^{(k)}})_j} = \sum_{j\in S^{(k+1)}} q_j \left(\log u_j - \log \left(\sum_{i\in S^{(k)}} u_i \Pi(i,j) - \gamma \sum_{i \in S^{(k)}\setminus S^{(k+1)}} u_i \Pi(i,j) \right)\right)$$

This is a non decreasing function of γ, that tends to the value

$$\sum_{j\in S^{(k)}} q_j \log \frac{u'_j}{(u'\Pi_{S^{(k)}})_j} \xrightarrow[\gamma\to 1]{} \sum_{j\in S^{(k+1)}} q_j \log \frac{u_j}{(u\Pi_{S^{(k+1)}})_j}$$

when $\gamma \to 1$. Thus,

$$J^{(k+1)}(q) = \sup_{u>0} \sum_{j\in S^{(k+1)}} q_j \log \frac{u_j}{(u\Pi_{S^{(k+1)}})_j} = \sup_{u>0} \sum_{j\in S^{(k+1)}} q_j \log \frac{u_j}{(u\Pi_{S^{(k)}})_j} = J^{(k)}(q). \quad (3.18)$$

Now it remains to actually run the recurrence. Consider the Markov chain $\left(X_n^{(k)}\right)$ defined over $S^{(k)}$ by the transition kernel $\Pi_{S^{(k)}}$.

If $S^{(k)} = S^{(k+1)}$, that is to say if every state in $S^{(k)}$ has an arrow pointing toward them in the graph of transitions of the Markov chain $\left(X_n^{(k)}\right)$, then by Lemma 3.4, $I^{(k)}(q) = J^{(k)}(q)$. The recursive process stops at k. If $S_q \not\subset S^{(k+1)}$, then by lemma 3.5, $I^{(k)}(q) = J^{(k)}(q) = +\infty$. The recursive process stops at k.

Else, we consider $\left(X_n^{(k+1)}\right)$ defined over $S^{(k+1)}$ by the transition kernel $\Pi_{S^{(k+1)}}$. The cardinal of $S^{(k+1)}$ is strictly lower than the cardinal of $S^{(k)}$. We apply the same reasoning to $\left(X_n^{(k+1)}\right)$.

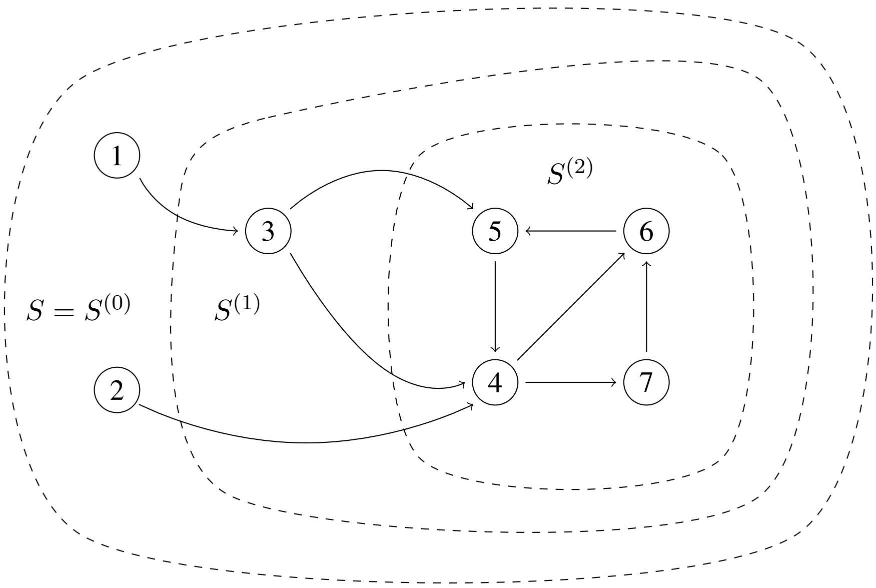
Eventually, this recursive process will stop because $S = S^{(0)}$ is finite, and the size of $S^{(k)}$ cannot decrease infinitely many times. When it stops, at some $k \geq 0$, either $S_q \subset S^{(k+1)}$ or $S^{(k)} = S^{(k+1)}$. In both cases, the arguments above are in force and one can conclude that $I^{(k+1)}(q) = J^{(k+1)}(q)$. Thus

$$I(q) = I^{(1)}(q) = \ldots = I^{(k+1)}(q) = J^{(k+1)}(q) = \ldots = J^{(1)}(q) = J(q).$$

This completes the proof. □

Remark 3.6. Notice that under **(Irr)** assumption, one can verify directly that $S' = S$ so the proof does not need the recurrence argument. See Remarks 4.17 and 4.18 below for this simpler proof.

Remark 3.7. We already discussed the meaning of the definition of $S^{(k)}$. Observe that restraining from $S^{(k)}$ to $S^{(k+1)}$ is removing the states of the Markov chain $\left(X_n^{(k)}\right)$ which can only occur at time 1. Restraining from S to $S^{(k+1)}$ is thus removing the states of (X_n) that can only occur at times lower that $k+1$. After a deterministically finite time, they will never be reached again by X_n. Such states are meaningless in a large deviation point of view, because for n large enough the probability for L_n^X to charge them more than a positive constant is always zero. This is the sense of Lemma 3.5. The following example illustrates the definition of $S^{(k)}$ and the need to use it.



3.3 Equality of K and L

Let us only assume that Π is irreducible and drop the assumption $\Pi > 0$. In the following we show that $L = K$ by arguments based on the Legendre-Fenchel transform. Arguments are adapted from Theorems 13.1 and 13.2 in [RAS15] to the finite-dimensional case.

Proposition 3.8. *For all $q \in \mathcal{M}_1(S)$, $K(q) = L(q)$.*

We provide here only a partial proof of Proposition 3.8. Some technical arguments are to be found in the proof of [RAS15, Theorem 13.1].

Proof. Let Λ_1 and Λ_2 be defined respectively on $\mathbb{R}^S$ and $\mathbb{R}^S \times \mathbb{R}^S$ as follow:

$$\Lambda_1(w) = \sum_i q_i \log\left(\sum_j \Pi(i,j)e^{w_j}\right), \tag{3.19}$$

$$\Lambda_2(w) = \sum_i q_i \log\left(\sum_j \Pi(i,j)e^{w_{i,j}}\right). \tag{3.20}$$

When $w \in \mathbb{R}^S \times \mathbb{R}^S$ does not depend on its first coordinate, we sure have $\Lambda_2(w) = \Lambda_1(w)$. Both functions are lower semicontinuous and convex by Hölder inequality. Notice that Λ_1 is fairly linked to K:

$$\begin{aligned} K(q) &= \sup_{v>0} \sum_i \alpha_i \log \frac{v_i}{(\Pi v)_i} \\ &= \sup_{w \in \mathbb{R}^{S_q}} \sum_i q_i \left(w_i - \log(\Pi e^w)_i\right) \\ &= \sup_{w \in \mathbb{R}^{S_q}} \left(\langle q, w\rangle - \Lambda_1(w)\right) & (3.21) \\ &= \Lambda_1^*(q). & (3.22) \end{aligned}$$

Remark 3.9. As Λ_1 has q as a parameter, the right-hand function of q in line (3.21) is not the Legendre-Fenchel transform of Λ_1. However, if q is fixed and r is a measure taken as a variable, $\sup_{w\in\mathbb{R}^{S_q}}(\langle r,w\rangle - \Lambda_1(w))$ really is the transform of Λ_1 at r, and it is possible to evaluate it at $r=q$. Thus it is actually true that $K(q)=\lambda_1^*(q)$.

Let $w\in\mathbb{R}^S$. By Fenchel-Moreau Theorem (see [Bre99] for a reference on the Fenchel-Moreau Theorem), Λ_2 is its convex biconjugate, so

$$\Lambda_1(w)=\Lambda_2(w)=\Lambda_2^{**}(w)=\sup_{\nu\in\mathcal{M}_1(S\times S)}(\langle\nu,w\rangle-\Lambda_2^*(\nu)). \tag{3.23}$$

The actual dual of $\mathbb{R}^S\times\mathbb{R}^S$ is also $\mathbb{R}^S\times\mathbb{R}^S$, but $\Lambda_2^*(\nu)$ is infinite whenever ν is not a probability measure, so the supremum can be taken only on $\mathcal{M}_1(S\times S)$. Indeed, if there exists $(i_0,j_0)\in S\times S$ such that $\nu(i_0,j_0)<0$, with $w=(\delta_{i,i_0}\delta_{j,j_0})\in\mathbb{R}^S\times\mathbb{R}^S$ and $c>0$,

$$\begin{aligned}\Lambda_2^*(\nu)&\geq\langle\nu,-cw\rangle-\Lambda_2(-cw)\\&=-c\nu(i_0,j_0)+cq_{i_0}-\sum_i q_i\log\Pi(i,j_0)\xrightarrow[c\to+\infty]{}+\infty.\end{aligned}$$

If $\nu(S\times S)>1$, for $w_{i,j}=c$,

$$\begin{aligned}\Lambda_2^*(\nu)&\geq\langle\nu,w\rangle-\Lambda_2(w)\\&=-c\nu(S\times S)-c\sum_i q_i=c(\nu(S\times S)-1)\xrightarrow[c\to+\infty]{}+\infty.\end{aligned}$$

This shows equation (3.23). As w does not depend on the second coordinate, $\langle\nu,w\rangle=\langle\nu_1,w\rangle$ where ν_1 denotes the first marginal of ν. Thus we derive

$$\begin{aligned}\Lambda_1(w)&=\sup_{\nu\in\mathcal{M}_1(S\times S)}(\langle\nu,w\rangle-\Lambda_2^*(\nu))\\&=\sup_{\nu\in\mathcal{M}_1(S\times S)}(\langle\nu_1,w\rangle-\Lambda_2^*(\nu))\\&=\sup_{r\in\mathcal{M}_1(S)}\left(\sup_{\substack{\nu\in\mathcal{M}_1(S\times S)\\\nu_1=r}}(\langle\nu_1,w\rangle-\Lambda_2^*(\nu))\right)\\&=\sup_{r\in\mathcal{M}_1(S)}\left(\langle r,w\rangle-\inf_{\substack{\nu\in\mathcal{M}_1(S\times S)\\\nu_1=r}}\Lambda_2^*(\nu)\right).\end{aligned}\tag{3.24}$$

This equality holds for every $w\in\mathbb{R}^S$, thus if M denotes the function $\mathcal{M}_1(S)\ni r\mapsto\inf_{\nu,\nu_1=r}\Lambda_2^*(\nu)$, we just showed that $M^*=\Lambda_1$.

By Fenchel-Moreau Theorem, $M=M^{**}=\Lambda_1^*$ over $\mathcal{M}_1(S)$. It means that for every probability measure r,

$$M(r)=\sup_{w\in\mathbb{R}^{S_q}}(\langle r,w\rangle-\Lambda_1(w)) \tag{3.25}$$

In particular, for $r=q$, it yields $M(q)=K(q)$ in virtue of equation (3.22). Now it remains to show that $M(q)=L(q)$. To do so, the proof of [RAS15, Theorem 13.1] yields that for any $\nu\in\mathcal{M}_1(S\times S)$, if $\nu_2=q$, then

$$\Lambda_2^*(\nu)=H(\nu(i,j)|q_i\Pi(i,j)) \tag{3.26}$$

and $\Lambda_2^*(\nu) = \infty$ else. Therefore,

$$\begin{aligned} M(q) &= \inf_{\nu_1 = q} \Lambda_2^*(\nu) \\ &= \inf_{\substack{\nu_1 = q \\ \nu_2 = q}} \sum_{i,j} \nu(i,j) \log \frac{\nu(i,j)}{q_i \Pi(i,j)} \quad \text{(notice that this function is infinite if } \nu \not\ll (q_i \Pi(i,j))_{i,j}) \\ &= \inf_{Q \in \mathcal{K}_S(q)} \sum_{i,j} q_i Q(i,j) \log \frac{q_i Q(i,j)}{q_i \Pi(i,j)} = L(q). \end{aligned}$$

The last line is possible by taking $q_i Q(i,j) = \nu(i,j)$ for every ν, which defines a stochastic kernel stabilizing q if $\nu_1 = \nu_2 = q$ (conversely, defining ν from Q gives a measure whose both marginals are q). We have proved that $K(q) = M(q) = L(q)$. □

4 The irreducible case: optimization of the variational formulae

The previous section showed that $I = J = K = L$ but did not express the links between the functions f_I^q, f_J^q, f_K^q and f_L^q. Under **(Irr)** and a fortiori under **(Pos)**, some stronger relations are satisfied and enlighten the links between I, J, K, and L. In this section, we will discuss the existence or not of optimizers for f_I^q, f_J^q, f_K^q and f_L^q, and find relations between them.

4.1 Maximizing f_K^q

We wonder whether there exists a maximizer of f_K^q. We will distinguish two cases in the following proposition and corollary.

Proposition 4.1. *If $S_q \neq S$, then f_K^q does not reach its supremum. However, under* **(Pos)**,

$$K(q) = \sup_{v>0} f_K^q = \sup_{v>0} \sum_{i \in S_q} q_i \log \frac{v_i}{\sum_{j \in S_q} \Pi(i,j) v_j}, \tag{4.1}$$

and the right-hand supremum is reached.

Corollary 4.2. *Under* **(Pos)**, *if $S_q = S$, then f_K^q has a maximizer.*

Corollary 4.2 is a direct consequence of Proposition 4.1 because f_K^q is the optimized function of the righ-hand term in (4.1) when $S_q = S$.

Remark 4.3. When q has some null coordinates, we will understand that f_K^q cannot reach its maximum because forcing some coordinates of v to get closer to 0 improves the value of $f_K^q(v)$. Thus, the right-hand supremum in (4.1) is only the limit case, when we allow v to have null coordinates.

Proof of Proposition 4.1. Assume $S_q \neq S$. Let $\gamma \in [0,1)$. For any $v > 0$, consider a new vector $v' > 0$ with coordinates

$$v'_i = \begin{cases} v_i & \text{if} \quad i \in S_q \\ (1-\gamma) v_i & \text{if} \quad i \notin S_q. \end{cases}$$

Then,

$$f_K^q(v') = \sum_{i\in S_q} q_i \left(\log v_i - \log\left(\sum_{j\in S}\Pi(i,j)v_j - \gamma\sum_{j\notin S_q}\Pi(i,j)v_j\right)\right).$$

This expression is (strictly) increasing with γ, and $f_K^q(v') = f_K^q(v)$ when $\gamma = 0$. Thus there is no maximizer of f_K^q because one could always find a better v. Notice that the limit of $f_K^q(v')$ when $\gamma \to 1$ is

$$\lim_{\gamma\to 1} f_K^q(v') = \sum_{i\in S_q} q_i \left(\log v_i - \log\left(\sum_{j\in S_q}\Pi(i,j)v_j\right)\right) =: g_K^q(v). \tag{4.2}$$

The function g_K^q is the optimized function in the right-hand term in (4.1). Clearly,

$$\sup_{v>0} f_K^q(v) = \sup_{v'>0} f_K^q(v') = \sup_{v>0} g_K^q(v). \tag{4.3}$$

Now we want to show that g_K^q reaches its supremum. To do so, we will show that the research of such a maximizer can be restricted to a compact.

First, notice that for $i \notin S_q$, $g_K^q(v)$ is a constant function of v_i. Thus we can restrict the search of a maximizer to vectors having coordinate $v_i = 1$ for $i \notin S_q$. Moreover, multiplying v by a positive scalar does not change the value of $g_K^q(v)$, thus we can restrict the search of a maximizer to vectors such that $\min_{i\in S_q} v_i = 1$. As $g_K^q(1,\dots,1) \geq 0$, a maximizer of g_K^q should belong to the non-empty set $(g_K^q)^{-1}([0,\infty))$.

Consider some v in this set, such that $\min_{i\in S_q} v_i = v_{i_1} = 1$ and that $i \notin S_q \Rightarrow v_i = 1$. We are going to find an upper bound over its coordinates (of course, independent of v). Choose any index $i_2 \in S_q$, then we define an application k by $k(i_1) = i_2$ and $k(i) = i$ for any other index i. A sum of non-negative terms is greater than any of its terms, so

$$\begin{aligned}
g_K^q(v) &= \sum_{i\in S_q} q_i \left(\log v_i - \log \sum_{j\in S_q}\Pi(i,j)v_j\right)\\
&\leq \sum_{i\in S_q} q_i \Big(\log v_i - \log\left(\Pi(i,k(i))v_{k(i)}\right)\Big)\\
&= \sum_{i\in S_q} q_i(\log v_i - \log v_{k(i)}) - \sum_{i\in S_q} q_i \log\left(\Pi(i,k(i))\right)\\
&= q_{i_1}(\log v_{i_1} - \log v_{i_2}) - \sum_{i\in S_q} q_i \log\left(\Pi(i,k(i))\right).
\end{aligned} \tag{4.4}$$

As $\Pi(i,k(i)) \geq m = \min_{i,j}\Pi(i,j) > 0$, this inequality yields

$$0 \leq g_K^q(v) \leq q_{i_1}(\log v_{i_1} - \log v_{i_2}) - \log(m).$$

Recall that we defined i_1 for v_{i_1} to be 1, so we are able to finally get $0 \leq \log v_{i_2} \leq -\log(m)/q_{i_1}$ for any i_2. Thus, any coordinate of v is either 1 or in $[1, \exp(-\log(m)/q_{i_1})]$, and v lies in a compact set. As g_K^q is continuous, it has a maximizer over this compact set, and therefore it has a maximizer over $\{v, v > 0\}$. □

Remark 4.4. The assumption **(Pos)** was used to get $m > 0$. Under only **(Irr)**, some terms of the sum in (4.4) might be infinite and the domination does not work anymore. The function g_K^q can reach its supremum or not. Remark 4.8 below provides examples for both cases.

Now, in order to compute $K(q)$, we can begin to actually search for a maximizer of g_K^q. In the following, as g_K^q stays unchanged by any modification of v_i for $i \notin S_q$, we project $\mathbb{R}^S$ onto the vector space $\mathbb{R}^{S_q} = \mathbb{R}^{|S_q|}$, with an extracted matrix Π_{S_q} of Π. Recall the matrix multiplication defined previously:

$$(\Pi_{S_q} v)_i = \sum_{j \in S_q} \Pi(i,j) v_j.$$

Proposition 4.5. *Under* **(Pos)**, g_K^q *has a maximizer in* $\{v \in \mathbb{R}^{S_q}, v > 0\}$, *unique up to multiplication by a scalar, and characterized by the equivalence*

$$g_K^q \text{ reaches its maximum at } v \text{ if and only if } \frac{q_k}{v_k} = \sum_{i \in S_q} q_i \frac{\Pi(i,k)}{(\Pi_{S_q} v)_i} \text{ for all } k \in S_q. \tag{4.5}$$

The maximum of g_K^q is denoted by v^*, and is defined up to a scalar multiplication. By definition and equation (4.1), $K(q) = g_K^q(v^*)$.

Remark 4.6. Notice that (4.5) is actually an invariance condition on q. Denoting Q the stochastic kernel on S_q defined by $Q(i,j) = \Pi_{S_q}(i,j)v_j/(\Pi_{S_q} v)_i$, the equivalence (4.5) means that g_K^q reaches its maximum at v if and only if q is invariant under Q.

Remark 4.7. If $S_q = S$, then $f_K^q = g_K^q$ and Proposition 4.5 yields a characterisation of the maximizer of f_K^q, defined uniquely up to a scalar multiplication. In the following proof, it is interesting to consider the case $S_q = S$. Actually, note that most of the time the measure considered will be such that $S_q = S$.

Proof of Proposition 4.5. Let $h_K^q(w) = g_K^q(e^w)$ defined of $\mathbb{R}^{S_q}$, so that h_K^q and g_K^q share their supremum, and $g_K^q(v)$ is maximal if and only if $h_K^q(\log v)$ is. Notice that adding a constant (*i.e.* a vector all of whose coordinates are the same) to the argument is like multiplying the argument of g_K^q by a scalar and does not change the value of h_K^q. The function h_K^q has an interesting property: its hessian matrix is semi-negative-definite everywhere. Let us compute it.

$$h_K^q(w) = \sum_{i \in S_q} q_i \left(w_i - \log \left(\sum_{j \in S_q} \Pi(i,j) e^{w_j} \right) \right), \tag{4.6}$$

$$(\nabla h_K^q(w))_k = q_k - \sum_{i \in S_q} q_i \frac{\Pi(i,k) e^{w_k}}{(\Pi_{S_q} e^w)_i}, \tag{4.7}$$

$$(H h_K^q(w))_{k,l} = - \sum_{i \in S_q} q_i \Pi(i,k) e^{w_k} \left(\frac{\Pi(i,l) e^{w_l}}{(\Pi_{S_q} e^w)_i^2} - \delta_{kl} \frac{1}{(\Pi_{S_q} e^w)_i} \right). \tag{4.8}$$

Now, $H h_K^q(w)$ is always a semi-negative-definite matrix: for all $x \in \mathbb{R}^{S_q}$,

$$\begin{aligned} x^T H h_K^q(w) x &= \sum_{k,l} x_k x_l H h_K^q(w)_{k,l} \\ &= \sum_i q_i \left(\left(\sum_k \frac{\Pi(i,k) x_k e^{w_k}}{(\Pi_{S_q} e^w)_i} \right)^2 - \frac{\Pi(i,k) e^{w_k} x_k^2}{(\Pi_{S_q} e^w)_i} \right) \leq 0. \end{aligned} \tag{4.9}$$

The reason why $x^T H h_K^q(w) x$ is non-positive in (4.9) is Jensen inequality. Apply Jensen inequality to the square function and to the points x_k with coefficients $\Pi_{S_q}(i,k) e^{w_k} / (\Pi_{S_q} e^w)_i$ to find that each term in the sum in (4.9) is non-positive, therefore $x^T Hh_K^q(w)x$ is nonpositive, and $Hh_K^q(w)$ is a semi-negative-definite matrix.

This grants that any critical point of h_K^q is a local maximizer. Now we also show that it is a global maximizer.

Let w be a local maximizer of h_K^q. Let w' be any point of $\mathbb{R}^{S_q}$, and consider the function $\varphi(t) := h_K^q((1-t)w + tw')$. It satisfies

$$\varphi'(t) = \nabla h_K^q((1-t)w + tw')(w' - w),$$
$$\varphi''(t) = (w' - w)^T Hh_K^q((1-t)w + tw')(w' - w) \leq 0,$$

so φ' is non-increasing. As $\varphi'(0) = \nabla h_K^q(w)(w' - w) = 0$ because w is a critical point, we get $\varphi(1) \leq \varphi(0) = 0$. It means that $h_K^q(w') \leq h_K^q(w)$. w is a global maximizer for h_K^q.

We have shown that beeing a critical point of h_K^q is a sufficient condition to maximize h_K^q. With the previous expression for ∇h_K^q in equation (4.7), we get the equivalence (4.5):

$$g_K^q \text{ reaches its maximum at } v = e^w \text{ if and only if } \frac{q_k}{v_k} = \sum_{i \in S_q} q_i \frac{\Pi_{S_q}(i,k)}{(\Pi_{S_q} v)_i} \text{ for all } k \in S_q.$$

For the uniqueness, let w and w' be global maximizers, and consider again $\varphi(t) = h_K^q((1-t)w + tw')$. It satisfies $\varphi'(0) = \varphi'(1) = 0$. By Rolle Theorem its second derivative has to cancel out at some t, thus for this t, $\varphi''(t) = (w' - w)^T Hh_K^q((1-t)w + tw')(w' - w) = 0$. By Jensen inequality in the equation (4.9), this is impossible unless all the $w'_k - w_k$ are equal. This means $w'_k = w_k + c$ for all k. Therefore, up to a constant (here, c), w is the only critical points of h_K^q. Therefore it also is the only maximizer. Notice that this point relying on Jensen inequality used that every coefficient of Π_{S_q} is positive but not that every coeficient of Π is. □

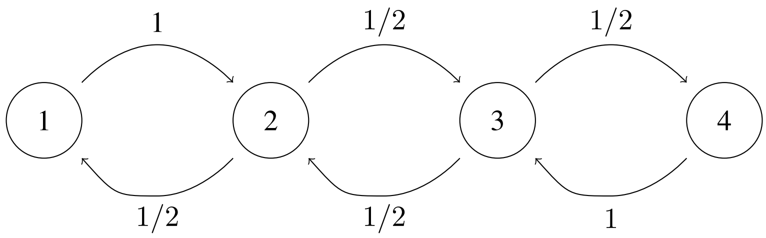
Remark 4.8. In the proof of equivalence (4.5), **(Pos)** was not actually fully used. One can weaken the assumption with

(Pos') $\forall i, j \in S_q \quad \Pi(i,j) > 0.$

While the equivalence (4.5) remains true under **(Pos')**, both the existence and uniqueness of a maximizer do not hold, in general. Let us see what happens under **(Pos')**. Remark 4.4 underlined that one cannot obtain existence in the same way than in the proof Proposition 4.1. For the uniqueness, the proof of Proposition 4.5 uses the equality case in Jensen inequality. If the coefficients of Π are no longer assumed positive, even if **(Pos')** constrains $(\Pi_{S_q} v)_i$ to be positive, one does not necessarily have $\Pi(i,k)e^{w_k}/(\Pi_{S_q} e^w)_i > 0$ for all k, and in particular it is possible that only one of those coefficients is positive and all the other are null, thus cancelling our chances to use the Jensen inequality. In our case, it would mean that the state i leads the Markov process to a certain state k with probability 1. For instance, consider the following transition matrix mentioned in [RAS15, Example 13.19]:

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.10}$$

where the states 1 and 4 lead to states 2 and 3 respectively with probability 1. It corresponds to the following graph:



Take $q = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Then, the function $g_K^q(v)$ is given by the formula

$$g_K^q(v) = \frac{1}{2}\log\frac{v_1}{v_2} + \frac{1}{2}\log\frac{2v_2}{v_1} = \frac{1}{2}\log 2.$$

It is a constant of v, and has several maximizers. Also notice that f_K^q has no maximizer at all:

$$f_K^q(v) = \frac{1}{2}\log\frac{v_1}{v_2} + \frac{1}{2}\log\frac{2v_2}{v_1 + v_3} = \frac{1}{2}\log 2 + \frac{1}{2}\log\frac{1}{1 + v_3/v_1}.$$

We knew that f_K^q couldn't have maximizers because $S_q \neq S$. The same Markov chain also yields an example where the supremum of g_K^q is not attained, with $q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. The value of $g_K^q(v) = f_K^q(v)$ is given by the expression

$$f_K^q(v) = \frac{1}{2}\log 2 - \frac{1}{4}\log\left((1 + \frac{v_3}{v_1})(1 + \frac{v_2}{v_4})\right),$$

that becomes greater and greater when $\frac{v_3}{v_1}$ or $\frac{v_2}{v_4}$ tends to zero.

4.2 Maximizing f_J^q

One can rewrite the entire previous section with left matrix multiplication instead of right matrix multiplication. Define g_J^q by

$$g_J^q(u) = \sum_{j \in S_q} q_j \left(\log u_j - \log\left(\sum_{i \in S_q} u_i \Pi(i,j)\right)\right). \tag{4.11}$$

Proposition 4.9. *Under* **(Pos)**, $J(q) = \sup_{u>0} f_J^q(u) = \sup_{u>0} g_J^q(u)$, *and* g_J^q *has a maximizer in* $\{u \in \mathbb{R}^{S_q}, u > 0\}$, *unique up to multiplication by a scalar, and characterized by the equivalence*

$$g_J^q \text{ reaches its maximum at } u \text{ if and only if } \frac{q_k}{u_k} = \sum_{j \in S_q} q_j \frac{\Pi(k,j)}{(u\Pi_{S_q})_j} \text{ for all } k \in S_q. \tag{4.12}$$

To prove this statement, simply repeat the previous section. The maximum of g_J^q, up to its degeneration is denoted u^*. By definiton, $g_J^q(u^*) = J(q)$. Previous remarks 4.3, 4.4, and 4.7 hold also for J.

4.3 Minimizing f_L^q

In order to understand better how f_L^q can be minimized, we start by restraining its domain to stochastic kernels over S_q. Like in the previous sections, we project the space $\mathbb{R}^S$ on $\mathbb{R}^{S_q}$. As we will see, this is natural, because when Q is a stochastic kernel over S that stabilizes q, its lines and columns of index outside S_q are meaningless in the computation of $f_L^q(Q)$. For this reason, Q could be considered only over S_q. We will actually see that the extracted matrix Q_{S_q} is a stochastic kernel. In the following, we work under **(Pos)** assumption.

Proposition 4.10. *Let g_L^q be the function defined over $\mathcal{K}_{S_q}(q)$ (the set of stochastic kernels of S_q that stabilize q) by*

$$g_L^q(Q) = \sum_{i,j \in S_q} q_i Q(i,j) \log \frac{Q(i,j)}{\Pi(i,j)}. \tag{4.13}$$

Then, under **(Pos)**, *g_L^q and f_L^q reach their respective infima and*

$$L(q) = \inf_{Q \in \mathcal{K}_S(q)} f_L^q(Q) = \inf_{Q' \in \mathcal{K}_{S_q}(q)} g_L^q(Q'). \tag{4.14}$$

Remark 4.11. When $S_q = S$, that is to say most of the time, Proposition 4.10 is trivial because $g_L^q = f_L^q$. in the following proof, we focus on $S \backslash S_q$.

Proof of Proposition 4.10. The function f_L^q is continuous over $\mathcal{K}_S(q)$ that is compact as the dimension is finite. Thus it has a minimizer. Same for g_L^q with the compactness of $\mathcal{K}_{S_q}(q)$.

Let Q be a minimizer of f_L^q, and simply define $Q'(i,j) = Q(i,j)$ for $i,j \in S_q$ (that is to say Q' is the extracted matrix Q_{S_q}). We want to show that $Q' \in \mathcal{K}_{S_q}(q)$ and that $g_L^q(Q') = f_L^q(Q)$. First, notice that by the invariance condition, for every $j \notin S_q$,

$$0 = q_j = (qQ)_j = \sum_{i \in S} q_i Q(i,j) = \sum_{i \in S_q} q_i Q(i,j).$$

As $q_i > 0$ for $i \in S_q$, the coefficients $Q(i,j)$ must be null when $i \in S_q$ and $j \notin S_q$. Thus, as Q is a stochastic kernel over S, for all $i \in S_q$,

$$\sum_{j \in S_q} Q'(i,j) = \sum_{j \in S_q} Q(i,j) = \sum_{j \in S} Q(i,j) = 1,$$

and Q' is a stochastic kernel over S_q. Second, by the invariance condition for Q, for all $j \in S_q$,

$$(qQ')_j = \sum_{i \in S_q} q_i Q(i,j) = \sum_{i \in S} q_i Q(i,j) = (qQ)_j = q_j.$$

Therefore, $Q' \in \mathcal{K}_{S_q}(q)$. Finally, by removing null terms in the expression of $f_L^q(Q)$,

$$\begin{aligned}
\inf_{\mathcal{K}_S(q)} f_L^q = f_L^q(Q) &= \sum_{i \in S} \sum_{j \in S} q_i Q(i,j) \log \frac{Q(i,j)}{\Pi(i,j)} \\
&= \sum_{i \in S_q} \sum_{j \in S} q_i Q(i,j) \log \frac{Q(i,j)}{\Pi(i,j)} \\
&= \sum_{i \in S_q} \sum_{j \in S_q} q_i Q(i,j) \log \frac{Q(i,j)}{\Pi(i,j)} = g_L^q(Q') \geq \inf_{\mathcal{K}_{S_q}(q)} g_L^q.
\end{aligned}$$

Now we want to prove the converse inequality. Let $Q' \in \mathcal{K}_{S_q}(q)$ be a minimizer of g_L^q. We can find $Q \in \mathcal{K}_S(q)$ such that $f_L^q(Q) = g_L^q(Q')$. Indeed, let

$$Q(i,j) = \begin{cases} \delta_{ij} & \text{if } i \notin S_q \\ Q'(i,j) & \text{if } i \in S_q, j \in S_q \\ 0 & \text{else.} \end{cases}$$

This is basically extending Q' over S by saying it should be the identity over $S\backslash S_q$. One can easily check that $Q \in \mathcal{K}_S(q)$. Moreover, by removing null terms in the expression of $f_L^q(Q)$,

$$\begin{aligned} f_L^q(Q) &= \sum_{i\in S}\sum_{j\in S} q_i Q(i,j)\log\frac{Q(i,j)}{\Pi(i,j)} \\ &= \sum_{i\in S_q}\sum_{j\in S_q} q_i Q(i,j)\log\frac{Q(i,j)}{\Pi(i,j)} \\ &= \sum_{i\in S_q}\sum_{j\in S_q} q_i Q'(i,j)\log\frac{Q'(i,j)}{\Pi(i,j)} = g_L^q(Q') = \inf_{\mathcal{K}_{S_q}(q)} g_L^q. \end{aligned}$$

This shows $\inf_{\mathcal{K}_S(q)} f_L^q \leq \inf_{\mathcal{K}_{S_q}(q)} g_L^q$, completing the proof. □

Once again, minimizing g_L^q is an easier task than minimizing f_L^q. One can get some explicit necessary conditions for minimizers of g_L^q using the Lagrange multipliers method.[2] Recall that under **(Pos)**, v^* is defined as the unique (up to scalar multiplication) maximizer of g_K^q.

Proposition 4.12. *Assume* **(Pos)**. *Then the minimizer of g_L^q is uniquely defined by the expression*

$$Q^*(i,j) = \Pi_{S_q}(i,j)\frac{v_j^*}{(\Pi_{S_q}v^*)_i}, \tag{4.15}$$

for $i,j \in S_q$.

Remark 4.13. If $S_q \neq S$, one could extend this kernel to S with the identity over $S\backslash S_q$ and get a stochastic kernel over S that minimizes f_L^q. The uniqueness of the minimizer is lost though, because the lines of index $i \notin S_q$ could be replaced by any non-negative line that sums up to 1 without changing the value of f_L^q.

Proof of Proposition 4.12. Let us find necessary conditions on the minimizer of

$$\phi : x \mapsto \sum_{i,j\in S_q} q_i x_{ij}\log\frac{x_{ij}}{\Pi(i,j)},$$

over $[0,\infty)^{S_q\times S_q}$ with the following $2|S_q|$ constraints

1. $\forall j \in S_q \quad \sum_{i\in S_q} q_i x_{ij} = q_j$ (q is invariant by Q),
2. $\forall i \in S_q \quad \sum_{j\in S_q} x_{ij} = 1$ (each line of Q sums up to 1).

We use the Lagrange multipliers method. Let

$$\Phi(x,\lambda,\nu) := \phi(x) + \sum_{j\in S_q}\lambda_j\left(\sum_{i\in S_q} q_i x_{ij} - q_j\right) + \sum_{i\in S_q}\nu_i\left(\sum_{j\in S_q q} x_{ij} - 1\right). \tag{4.16}$$

A minimizer x of ϕ with these two constraints satisfies $\nabla\Phi(x,\lambda,\nu) = 0$ for some $(\lambda,\nu) \in \mathbb{R}^{S_q}\times\mathbb{R}^{S_q}$. One has

$$\frac{\partial\Phi}{\partial x_{ij}} = q_i\left(1+\log\frac{x_{ij}}{\Pi(i,j)}\right) + \lambda_j q_i + \nu_i,$$

[2] This is not the usual method to do so, because it is specific to the finite dimension. However, it is quite efficient.

and as the partial derivative has to cancel out at (x,λ,ν), we get $x_{ij}=\Pi(i,j)\exp(-\frac{\nu_i}{q_i}-\lambda_i-1)$. This yields the existence of some $a,b\in(0,\infty)^{S_q}$ such that $x_{ij}=\Pi(i,j)a_ib_j$. The constraint 2 implies that $a_i=(\Pi_{S_q}b)_i^{-1}$ for every i, so that $x_{ij}=\Pi(i,j)\frac{b_j}{(\Pi_{S_q}b)_i}$. Then the constraint 1 yields

$$\forall j\in S_q\quad \frac{q_j}{b_j}=\sum_{i\in S_q}q_i\Pi(i,j)\frac{1}{(\Pi_{S_q}b)_i}\tag{4.17}$$

for every j. Notice that the set of equations (4.17) is actually the right-hand side of equivalence (4.5). It means that, up to a multiplication by a scalar, $b=v^*$. Therefore, x satisfies

$$\forall i,j\in S_q\quad x_{ij}=\Pi(i,j)\frac{v_j^*}{(\Pi_{S_q}v^*)_i}.\tag{4.18}$$

The minimizer x is thus unique. This completes the proof because the set of x that satisfies constaints 1 and 2 is the set $\mathcal{K}_{S_q}(q)$ and the restriction of ϕ to $\mathcal{K}_{S_q}(q)$ is g_L^q. □

Remark 4.14. It is not possible to copy this reasoning with u^* instead of v^*, because the computation of Q^* imposes the right multiplication. The equation (4.17) comes without any choice and favors the right matrix multiplication.

Remark 4.15. When **(Pos)** is removed and replaced by **(Irr)**, one should only consider the stochastic kernels of $\mathcal{K}_{S_q}(q)$ that are absolutely continuous with respect to Π_{S_q}, otherwise $g_L^q(Q)=+\infty$.

The already discussed example of a 4-state Markov chain defined by (4.10) is interesting. For $q=(\frac14,\frac14,\frac14,\frac14)$, the only kernel which is absolutely continuous with respect to Π and stabilizes q is

$$Q=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

thus it is the one to maximize f_L^q. Another way to see it is that Q corresponds to the limit case of $Q(i,j):=\frac{\Pi(i,j)v_j}{(\Pi v)_i}$ when v_3 and v_2 both tend to 0, which was the correct limit to consider in order for $f_K^q(v)$ to approach its supremum.

Remark 4.16. Thanks to Proposition 4.12, the relation between Q^* and v^* provides a simple proof of Proposition 3.8 under additional assumption **(Pos)**. Let us detail its computations.

Recall Q^* and v^* are only defined over S_q. As q is invariant under Q^*,

$$\begin{aligned}K(q)=g_K^q(v^*)&=\sum_{j\in S_q}q_j\log\frac{v_j^*}{(\Pi_{S_q}v^*)_j}\\&=\sum_{j\in S_q}\sum_{i\in S_q}q_iQ^*(i,j)\log\frac{v_j^*}{(\Pi_{S_q}v^*)_j}\\&=\sum_{j\in S_q}\sum_{i\in S_q}q_iQ^*(i,j)\left(\log\frac{Q^*(i,j)}{\Pi_{S_q}(i,j)}+\log\frac{(\Pi_{S_q}v^*)_i}{(\Pi_{S_q}v^*)_j}\right)\\&=g_L^q(Q^*)+\alpha,\end{aligned}$$

with a remainder α :

$$\alpha=\sum_{i,j\in S_q}q_iQ^*(i,j)\log((\Pi_{S_q}v^*)_i)-\sum_{i,j\in S_q}q_iQ^*(i,j)\log((\Pi_{S_q}v^*)_j).$$

As the sum over the lines of Q^* has to be 1, the first term is actually $\langle q, \log(\Pi_{S_q} v^*)\rangle$, and as q is invariant under Q^*, the second is actually $-\langle q, \log(\Pi_{S_q} v^*)\rangle$. The remainder α vanishes! Finally,

$$K(q) = g^q_K(v^*) = g^q_L(Q^*) = L(q). \tag{4.19}$$

One can notice that this computation holds when v^* is replaced by any v and Q^* by the stochastic kernel $Q(i,j) = \frac{\Pi_{S_q}(i,j)v_j}{(\Pi_{S_q} v)_i}$, as long as Q keeps the property of stabilization of q.

4.4 Maximizing f^q_I

The computations carried out in the two following remarks will be useful in the search of a maximizer of f^q_I.

Remark 4.17. Proposition 3.2 already stated that that $I = K$. However, the assumption **(Irr)** widely simplifies its proof. Indeed, as Π is irreducible, the Perron-Frobenius Theorem provides a Perron-Frobenius eigenvector and an useful expression for $\rho(\Pi_\lambda)$. Let us see the details.

$\boxed{I(q) \le K(q).}$ Let $\lambda \in \mathbb{R}^{S_q}$. In the proof of Proposition 3.2, we had to define a vector v as the sum of a convergent series to get $f^q_I(\lambda) \le f^q_K(v)$. Now, we can get it easily thanks to Perron-Frobenius Theorem. Indeed, there exists $v > 0$ an eigenvctor of Π_λ associated with the eigenvalue $\rho(\Pi_\lambda)$. We have

$$\begin{aligned} f^q_I(\lambda) - f^q_K(v) &= \sum_{i\in S} q_i \log \frac{e^{\lambda_i}(\Pi v)_i}{v_i} - \log \rho(\Pi_\lambda) \\ &= \sum_{i\in S} q_i \log \frac{(\Pi_\lambda v)_i}{v_i} - \log \rho(\Pi_\lambda) \\ &= \sum_{i\in S} q_i \log \rho(\Pi_\lambda) - \log \rho(\Pi_\lambda) = 0. \end{aligned} \tag{4.20}$$

Thus we get $f^q_I(\lambda) = f^q_K(v) \le K(q)$. Taking the supremum over λ yields the inequality $I(q) \le K(q)$.

$\boxed{I(q) \ge K(q).}$ Let $v > 0$. Let λ be the vector of $\mathbb{R}^S$ of coordinates $\lambda_i = \log \frac{u_i}{(u\Pi)_j}$. Then, like in the proof of Proposition 3.2, $v = \Pi_\lambda v = \ldots = \Pi^n_\lambda v$. In the proof of Proposition 3.2, we used this eigenvector property to show that $\log \rho(\Pi_\lambda) \le 0$. But here, the Perron-Frobenius theorem provides an explicit expression, for any $i \in S$,

$$\log \rho(\Pi_\lambda) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{j\in S} \Pi^n_\lambda(i,j) v_j = \lim_{n\to\infty} \frac{1}{n} \log v_i = 0.$$

Thus, we get

$$I(q) = \sup_{\lambda \in \mathbb{R}^{S_q}} f^q_I(\lambda) \ge f^q_I(\lambda) = \langle \lambda, q\rangle - 0 = \sum_{i\in S} q_i \log \frac{v_i}{(\Pi v)_i} = f^q_K(v). \tag{4.21}$$

Taking the supremum over v yields $I(q) \ge \sup_{u>0} f^q_K(v) = J(q)$.

Remark 4.18. Assumption **(Irr)** not only brings the Perron-Frobenius Theorem, but also implies that $u\Pi > 0$ for $u > 0$, according to Remark 3.1. Thus, in the proof of Proposition 3.3, one gets $S' = S$ and it is possible to simply copy the arguments used for $I = K$ to get a simple proof of $I = J$. The arguments of the above remark 4.17 are also in force.

We can now begin to search a maximizer of f_I^q. First, notice that whenever a constant c is added to λ in the argument of f_I^q, one gets

$$f_I^q(c+\lambda) = \langle \lambda, q\rangle + c - \log\rho(\Pi_{\lambda+c}).$$

As $\Pi_{\lambda+c} = e^c\Pi_\lambda$ has as spectral radius of $e^c\rho(\Pi_\lambda)$,

$$f_I^q(c+\lambda) = \langle \lambda, q\rangle + c - (\log\rho(\Pi_\lambda) + c) = f_I^q(\lambda).$$

A vector λ that maximizes f_I^q would only be determined up to an additive constant. We do not know yet whether such a λ exists, *i.e.* whether f_I^q reaches its supremum.

Like in the previous sections, restraining the domain of Π to S_q allows us to find maximizers. For every $\lambda \in \mathbb{R}^{S_q}$, let

$$g_I^q(\lambda) := \langle \lambda, q\rangle - \log\rho(\Pi_{S_q}^\lambda) = \sum_{i\in S_q} \lambda_i q_i - \log\rho(\Pi_{S_q}^\lambda), \tag{4.22}$$

where $\Pi_{S_q}^\lambda$ is the matrix of coefficients $e^{\lambda_i}\Pi(i,j)$ for $i,j\in S_q$. Small changes in the proof in Remark 4.17 lead to the following statement. Recall v^* is the maximizer of g_K^q defined up to scalar multiplication.

Proposition 4.19. *Assume* **(Pos)** *is satisfied. Define* λ^* *by*

$$\forall i\in S_q \quad \lambda_i^* = \log\frac{v_i^*}{(\Pi_{S_q}v^*)_i}. \tag{4.23}$$

It is the only maximizer of g_I^q*, up to an additive constant, and it satisfies*

$$g_I^q(\lambda^*) = \sup_{\lambda\in\mathbb{R}^{S_q}} g_I^q(\lambda) = \sup_{\lambda\in\mathbb{R}^S} f_I^q(\lambda) = I(q). \tag{4.24}$$

If $S_q = S$, then $f_I^q = g_I^q$, implying the following corollary.

Corollary 4.20. *Assume* $S_q = S$ *and* **(Pos)** *are satisfied. Define* λ^* *by*

$$\forall i\in S \quad \lambda_i^* = \log\frac{v_i^*}{(\Pi v^*)_i}. \tag{4.25}$$

It is the only maximizer of f_I^q*, up to an additive constant.*

Remark 4.21. λ^* and q are actually in convex duality with respect to $\lambda\mapsto\log\rho(\Pi_\lambda)$.

Proof of Proposition 4.19. By Lemma 2.5, the function $\lambda\mapsto\log\rho(\Pi_{S_q}^\lambda)$ is strictly convex, thus g_I^q is stricly concave (in the sense of Lemma 2.5) and has at most one maximizer up to additive constant. We show that $\sup_{\lambda\in\mathbb{R}^{S_q}} g_I^q(\lambda) \le I(q)$. Let $\lambda\in\mathbb{R}^{S_q}$ and let v be an eigenvector of $\Pi_{S_q}^\lambda$ associated with the eigenvalue $\rho(\Pi_{S_q}^\lambda)$, which exists thanks to the Perron-Frobenius theorem. Like in the computation (4.20), one has

$$\begin{aligned}
g_I^q(\lambda) - g_K^q(v) &= \sum_{i\in S_q} q_i\lambda_i - \sum_{i\in S_q} q_i \log\frac{v_i}{(\Pi_{S_q}v)_i} - \log\rho(\Pi_{S_q}^\lambda)\\
&= \sum_{i\in S_q} q_i\log\frac{e^{\lambda_i}(\Pi_{S_q}v)_i}{v_i} - \log\rho(\Pi_{S_q}^\lambda)\\
&= \sum_{\in S_q} q_i\log\frac{(\Pi_{S_q}^\lambda v)_i}{v_i} - \log\rho(\Pi_{S_q}^\lambda)\\
&= \sum_{i\in S_q} q_i\log\rho(\Pi_{S_q}^\lambda) - \log\rho(\Pi_{S_q}^\lambda) \le 0.
\end{aligned} \tag{4.26}$$

It means $g_I^q(\lambda) \leq g_K^q(v) \leq K(q)$. As $I(q) = K(q)$ by Proposition 3.3, we just showed that $\sup_{\lambda \in \mathbb{R}^{S_q}} g_I^q(\lambda) \leq I(q)$. Now we can show that the bound $I(q)$ is reached to complete the proof. Under **(Pos)**, the maximizer v^* of g_K^q exists. Let (4.23) define λ^* over S_q. We have

$$(\Pi_{S_q}^{\lambda^*} v^*)_i = \sum_{j \in S_q} \Pi(i,j) \frac{v_i^*}{(\Pi_{S_q} v^*)_i} v_j^* = \frac{v_i^*}{(\Pi_{S_q} v^*)_i} (\Pi_{S_q} v^*)_i = v_i^*.$$

Thus v^* is an eigenvector of $\Pi_{S_q}^{\lambda^*}$ of eigenvalue 1. By the Perron-Frobenius theorem, it implies that

$$\log \rho(\Pi_{S_q}^{\lambda^*}) = \lim_{n \to \infty} \frac{1}{n} \log v_i = 0.$$

Therefore,

$$g_I^q(\lambda^*) = \langle \lambda^*, q \rangle = \sum_{i \in S_q} q_i \log \frac{v_i^*}{(\Pi_{S_q} v^*)_i} = g_K^q(v^*) = K(q) = I(q). \tag{4.27}$$

The maximum is reached. □

Remark 4.22. Once again, the study of g_I^q can be understood as the limit case in the study of f_I^q, where λ is allowed to have $-\infty$ coefficients. By convention, $0 \times \infty = 0$. Take $\lambda = -\infty$ over $S \backslash S_q$. Each line of index $i \notin S_q$ is a null line, thus Π_λ is no longer irreducible. However,

$$\Pi_\lambda^2(i,j) = \sum_{k \in S} \Pi_\lambda(i,k)\Pi_\lambda(k,j) = \sum_{k \in S_q} \Pi_\lambda(i,k)\Pi_\lambda(k,j).$$

When $i \notin S_q$, every term is null thus $\Pi_\lambda^2(i,j) = 0$ and by recurrence $\Pi_\lambda^n(i,j) = 0$, and when $i,j \in S_q$, this is exactly $(\Pi_{S_q}^\lambda)^2(i,j)$, so by recurrence $\Pi_\lambda^n(i,j) = (\Pi_{S_q}^\lambda)^n(i,j)$. Thus Π_λ^n behaves asymptotically like $(\Pi_{S_q}^\lambda)^n$, so their spectral radius should intuitively be the same. This says that $f_I^q(\lambda) = g_I^q(\lambda)$, and helps us understand the reason why g_I^q is useful here.

The previous reasonings holds when replacing the right multiplication by the left multiplication, thus comparing $f_I^q(\lambda)$ to $f_J^q(u)$ and $g_I^q(\lambda)$ to $g_J^q(u)$. It leads to the following conclusion.

Proposition 4.23. *Under* **(Pos)** *assumption, g_I^q has a unique maximizer λ^* up to additive constant, and the relation*

$$\forall i \in S \quad \lambda_i^* = \log \frac{u_i^*}{(u^*\Pi)_i} = \log \frac{v_i^*}{(\Pi v^*)_i}, \tag{4.28}$$

is satisfied up to additive constants.

4.5 The Perron-Frobenius point of view

Seeing the previous maximizers as Perron-Frobenius eigenvectors can be quite relevant in order to understand the links between the four rate functions. In the following, assume **(Irr)** is satisfied.

When λ is fixed, we denote by σ/s the left/right Perron-Frobenius eigenvector of Π_λ, and we fix one degenerescence in their definition by requiring $\langle \sigma, s \rangle = 1$. The vectors σ and s are positive, and point out a relation between f_I^q, f_J^q and f_K^q. Recall $D = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_N})$.

Lemma 4.24. *Let $q \in \mathcal{M}_1(S)$. Then $f_I^q(\lambda) = f_J^q(D\sigma) = f_K^q(s)$.*

Proof. This comes from

$$\begin{aligned}
f_K^q(s) &= \sum_i q_i \log \frac{s_i}{(\Pi s)_i} \\
&= \sum_i q_i \left(\lambda_i + \log \frac{s_i}{(\Pi_\lambda s)_i} \right) \\
&= \sum_i q_i \left(\lambda_i + \log \frac{s_i}{\rho(\Pi_\lambda) s_i} \right) \\
&= \langle q, \lambda \rangle - \log \rho(\Pi_\lambda) \\
&= f_I^q(\lambda).
\end{aligned} \tag{4.29}$$

A similar computation with left matrix multiplication and $D\sigma$ yields $f_J^q(D\sigma) = f_I^q(\lambda)$. □

Proposition 4.25. *Assume* **(Pos)** *and $S_q = S$ are satisfied. Let σ^*/s^* be the left/right Perron-Frobenius eigenvectors of Π_{λ^*}. Then, up to scalar multiplication $D\sigma^* = u^*$ and $s^* = v^*$.*

Proof. When $S_q = S$, v^* is the unique maximizer of $g_K^q = f_K^q$, up to scalar multiplication. In one hand, by Proposition 4.23, λ^* defined by $\lambda_i^* = \log \frac{v_i^*}{(\Pi v^*)_j}$ maximizes f_I^q, meaning that $f_I^q(\lambda^*) = I(q) = K(q)$. In the other hand, if s^* is the right Perron-Frobenius eigenvector of Π_{λ^*}, Lemma 4.24 yields that $f_K(s^*) = f_I^q(\lambda^*)$. Therefore $s^* = v^*$ up to scalar multiplication. Symmetrically, $D\sigma^* = u^*$ up to scalar multiplication. □

With the constraint $\langle \sigma, s \rangle = 1$, one could consider the probability measure q defined by $q_i = \sigma_i s_i$. It has all of its coordinates positive. An interesting fact is that for this specific q, maximizers of f_i^q, f_J^q, f_K^q, and f_L^q exist and are easily derived from σ and s.

Proposition 4.26. *Define a probability measure q by $q_i = \sigma_i s_i$. Then, under* **(Irr)***, f_i^q, f_J^q, f_K^q, and f_L^q reach their optima respectively at λ^*, u^*, v^*, and Q^* defined by*

$$\begin{cases}
\lambda_i^* = \lambda_i \\
u_i^* = e^{\lambda_i} \sigma_i \\
v_i^* = s_i \\
Q^*(i,j) = \frac{\Pi(i,j) s_j}{(\Pi s)_i}.
\end{cases} \tag{4.30}$$

Proof. Consider the stochastic kernel $Q(i,j) = \frac{\Pi(i,j)s_j}{(\Pi s)_i} = \frac{\Pi_\lambda(i,j)s_j}{\rho(\Pi_\lambda)s_i}$. It stabilizes q. Indeed, as $\Pi(i,j) = e^{-\lambda_i} \rho(\Pi_\lambda) Q(i,j) \frac{s_i}{s_j}$, the left eigenvector equation for Π_λ leads to

$$\rho(\Pi_\lambda)\sigma_j = \sum_i \sigma_i e^{\lambda_i} \Pi(i,j) = \sum_i \sigma_i \rho(\Pi_\lambda) Q(i,j) \frac{s_i}{s_j},$$

which reformulates after simplifications as $q_j = \sum_i Q(i,j) q_i$. Thus, according to remark 4.6 and equivalence (4.5), s maximizes f_K^q. v^* exists, and up to the degenerescences in the definitions of v^* and s, we have $s = v^*$. According to the equation (4.15), the kernel Q also minimizes f_L^q. Now thanks to Propositions 3.2 and 3.3 and Lemma 4.24,

$$I(q) = J(q) = K(q) = f_K^q(s) = f_J^q(D\sigma) = f_I^q(\lambda),$$

so $D\sigma$ maximizes f_J^q and λ maximizes f_I^q. □

In conclusion, when the probability vector q can be described as the coefficient-by-coefficeint product of the left/right Perron-Frobenius eigenvectors of a certain Π_λ, every extremum defining the four rate functions is attained, by quantities depending only on the eigenvectors, even without assuming Π positive.

5 Summary of the previous sections

Here we wrap up the previous sections in a few propositions.

Proposition 5.1. *If Π is a stochastic kernel, then $I = J = K = L$.*

Proposition 5.2. *Under assumption **(Irr)**, the LDP holds for L_n^X with rate function $I = J = K = L$.*

Proposition 5.3. *Under assumption **(Pos)**, the function $I = J = K = L$ is finite over $\mathcal{M}_1(S)$. The functions g_I^q, g_J^q, g_K^q, and g_L^q have unique optimizers λ^*, u^*, v^*, and Q^* respectively, which satisfy:*

$$\begin{cases} Q^*(i,j) = \Pi(i,j)\frac{v_j^*}{(\Pi_{S_q} v^*)_i} \\ \lambda_i^* = c + \log \frac{v_i^*}{(\Pi v^*)_i} = c' + \log \frac{u_i^*}{(u_i^* \Pi)_i} \end{cases} \tag{5.1}$$

Two more propositions helps us understand the optimizers and the links between them when q satisfies some constraints. Let $q \in \mathcal{M}_1(S)$.

Proposition 5.4. *Under assumptions **(Irr)** and $q = \sigma s$ for a certain λ, where σ and s are the left and right Perron-Frobenius eigenvectors of Π_λ, then $I(q)$, $J(q)$, $K(q)$, and $L(q)$ are finite, and f_i^q, f_J^q, f_K^q, and f_L^q reach their optima respectively at λ^*, u^*, v^*, and Q^* defined by*

$$\begin{cases} \lambda_i^* = \lambda_i \\ u_i^* = e^{\lambda_i} \sigma_i \\ v_i^* = s_i \\ Q^*(i,j) = \frac{\Pi(i,j) s_j}{(\Pi s)_i}. \end{cases} \tag{5.2}$$

Proposition 5.5. *Under assumptions **(Pos)** and $S_q = S$, the functions f_I^q, f_J^q, f_K^q, and f_L^q have unique optimizers λ^*, u^*, v^*, and Q^* respectively, which satisfy:*

$$\begin{cases} \lambda_i^* = c + \log \frac{v_i^*}{(\Pi v^*)_i} = c' + \log \frac{u_i^*}{(u_i^* \Pi)_i} \\ Q^*(i,j) = \Pi(i,j)\frac{v_j^*}{(\Pi v^*)_i} \\ v_i^* = s_i^* \\ u_i^* = e^{\lambda_i^*} \sigma_i^*. \end{cases} \tag{5.3}$$

6 Additional examples

6.1 Irreducible Markov chains on a two-states space

Let us consider a Markov chain X on a two-states space. Its transition matrix is denoted $\Pi := \begin{pmatrix} p & 1-p \\ 1-p' & p' \end{pmatrix}$, and the Markov chain is irreducible *iff* p and p' are strictly lower than 1. Then

Proposition 5.2 hold, and there is a LDP with rate function $I = J = K = L$. One may compute K over the set of probability vectors for $0 < p, p' < 1$.

Proposition 6.1. *If $0 < p, p' < 1$, then L_n^X satisfies a LDP with rate function $I = J = K = L$. Moreover, for q of positive coordinates,*

$$K(q) = q_1 \log \frac{1}{p + (1-p)\alpha} + q_2 \log \frac{\alpha}{(1-p') + p'\alpha},$$
$$\text{where } \alpha = \frac{(1-p)(1-p')(q_2 - q_1) + \sqrt{(1-p)^2(1-p')^2(q_1-q_2)^2 + 4q_1q_2pp'(1-p)(1-p')}}{2q_1(1-p)p'}. \tag{6.1}$$

If both p and p' are null the rate function derives from the expression of L: unless $q_1 = q_2 = \frac{1}{2}$, there is no stochastic kernel that stabilizes q while beeing absolutely continuous with respect to Π. Thus the infimum is $+\infty$, and the rate function is

$$L = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(\frac{1}{2},\frac{1}{2})\}}. \tag{6.2}$$

Proposition 6.2. *If $p = p' = 0$, then L_n^X satisfies a LDP with rate function $I = J = K = L = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(\frac{1}{2},\frac{1}{2})\}}$.*

Note that in this case one can obtain the LDP without Proposition 5.2. If both p and p' are null, then L_n^Y is a deterministic sequence that converge to $(\frac{1}{2}, \frac{1}{2})$ at rate $\frac{1}{n}$ once the initial state is known, so there is a LDP with rate function $\infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(\frac{1}{2},\frac{1}{2})\}}$.

The last irreducible case is when only one of them, say p', is null. We compute $L(q)$: a stochastic kernel Q absolutely continuous with respect to Π and stabilizing q can be written $Q = \begin{pmatrix} x & 1-x \\ 1 & 0 \end{pmatrix}$ with

$$\begin{cases} q_1 = xq_1 + q_2 \\ q_2 = (1-x)q_1. \end{cases}$$

Thus if $q_2 > q_1$ there is no such Q and $L(q) = +\infty$, and else there is only one Q determined by $x = 1 - \frac{q_2}{q_1}$, so

$$L(q) = f_L^q(Q) = -q_1 \log(q_1 p) + (q_1 - q_2)\log(q_1 - q_2) + q_2 \log(q_2).$$

Proposition 6.3. *If $p' = 0$, then then L_n^X satisfies a LDP with rate function $I = J = K = L$ with expression*

$$L(q) = -q_1 \log(q_1 p) + (q_1 - q_2)\log(q_1 - q_2) + q_2 \log(q_2). \tag{6.3}$$

6.2 The non-irreducible cases on a two-states space

In the reducible cases, Section 5 do not guarantee that the rate functions are equal, neither that a LDP holds. The only values of (p, p') that lead to a non-irreducible case are

- when $p = p' = 1$,
- when only one of them, say p', is equal to 1.

Start with the case $p = p' = 1$, that is to say $\Pi = \mathrm{id}_2$.

Proposition 6.4. *If $p = p' = 1$, then L_n^X satisfies a LDP with rate function I^{μ_0} depending on the initial measure μ_0.*

- *If μ_0 charges both points, then $I^{\mu_0} = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(1,0),(0,1)\}}$.*
- *If $\mu_0 = \delta_2$, then $I^{\mu_0} = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(0,1)\}}$.*
- *If $\mu_0 = \delta_1$, then $I^{\mu_0} = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(1,0)\}}$.*

Proof. The Markov chain X is actually deterministic and satisfies $\forall n \in \mathbb{N} \quad X_n = X_1$. Thus if μ_0 charges both states, L_n^X has values in $\{(1,0),(0,1)\}$ and its distribution does not change with n, so there is a LDP with rate function $I^{\mu_0} = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(1,0),(0,1)\}}$. If μ_0 only charges 1, then all X_n are 1 almost surely and $L_n^X = (1,0)$, so the LDP holds with rate function $I^{\delta_1} = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(0,1)\}}$. Same if it only charges 2, with rate function $I^{\delta_2} = \infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(1,0)\}}$. □

Remark 6.5. Let us compute I, J, K, and L, anyway. Start with $f_I^q(\lambda) = (\lambda_1 - \lambda_2)(q_1 \mathbb{1}_{\{\lambda_1 \le \lambda_2\}} - q_2 \mathbb{1}_{\{\lambda_2 \le \lambda_1\}})$, so f_I^q is maximized at $\lambda = 0$ with $f_I^q(0) = 0$, so $I(q) = 0$. the functions f_K^q and f_J^q are both null for any q, thus $K = J = 0$ uniformly. As for L, the only stochastic kernel absolutely continuous with respect to Π is $\Pi = \mathrm{id}_2$ itself. It sure stabilizes any measure q, so $L(q) = f_L^q(\mathrm{id}_2) = 0$. All four function are uniformly null over the set of probability measures. They are not equal to I^{μ_0}. Actually, they could not be associated with a LDP here, because the rate function of such a LDP has to be highly dependent the initial measure μ_0! The probability to be in state i at time n is $\mu_0(i)$ independently of n.

This case yields an example of a Markov chain for which rate functions I, J, K, L are not the ones associated with the LDP. It underlines that **(Irr)** is important for Proposition 5.2.

Now we consider the case of one transient state, with $\Pi = \begin{pmatrix} p & 1-p \\ 0 & 1 \end{pmatrix}$ and $0 < p < 1$.

Proposition 6.6. *If $p' = 0$ and $0 < p < 1$, and if μ_0 charges the first state 1, then L_n^X satisfies a LDP with rate function $q \mapsto -q_1 \log(p)$. If $p' = 0$ and $0 < p < 1$ and $\mu_0 = \delta_2$, then a LDP holds with rate function $\infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(0,1)\}}$.*

Proof. If $\mu_0(1) > 0$, the trajectory is entirely determined by the amount of time spent in the first state, which is geometrical. One has

$$\mathbb{P}\left(L_n^X = \left(\frac{k}{n}, \frac{1-k}{n}\right)\right) = \mu_0(1)p^{k-1}(1-p).$$

With $k_n = \lfloor nq_1 \rfloor$, we get

$$\mathbb{P}(L_n^X(1) \ge q_1) = \mathbb{P}\left(L_n^X(1) \ge \frac{k_n}{n}\right) = \sum_{i=k_n}^{\infty} \mu_0(1) \times p^{i-1}(1-p) = p^{\lfloor nq_1 \rfloor}\mu_0(1).$$

In an exponential scale, this means that L_n^X satisfies a LDP with rate function $q \mapsto -q_1 \log p$.

If μ_0 does not charge the first state, $L_n^X = (0,1)$ almost surely, and a LDP holds with rate function $\infty \times \mathbb{1}_{\mathcal{M}_1(S)\setminus\{(0,1)\}}$. □

Remark 6.7. Let us compute the four previous functions. The rate function I is given by:

$$f_I^q(\lambda) = \begin{cases} q_2(\lambda_2 - \lambda_1) - \log p & \text{if } \lambda_1 \ge \lambda_2 - \log p \\ q_1(\lambda_1 - \lambda_2) & \text{else} \end{cases} \qquad \le -q_1 \log p.$$

As $f_I^q((-\log p, 0)) = -q_1 \log p$, it reaches its maximum and $I(q) = -q_1 \log(p)$. The rate function K yields the same expression, because it is given by

$$f_K^q(v) = -q_1 \log\left(p + (1-p)\frac{v_2}{v_1}\right).$$

This has no maximizer but tends to its supremum when $\frac{v_2}{v_1} \to 0$, and so $K(q) = -q_1 \log(p)$. Similar computations also yield $J(q) = -q_1 \log(p)$.

As for L, if q is a probability measure, once again the only stochastic kernel that is simulaneously absolutely continuous with respect to Π and a stabilizator of q is id_2. Thus $L(q) = f_L^q(\mathrm{id}_2) = -q_1 \log(p)$.

Once again, the four rate functions are equal. But this time they actually are associated with the LDP for L_n^X, under the condition that μ_0 charges the first state. Even if **(Irr)** is not satisfied, the conclusion of Proposition 5.2 holds.

Remark 6.8. Notice that in the previous examples, even when I is not the rate function associated with the LDP, it is still its convex hull over the $\mathcal{M} - 1(q)$. This is a deep observation and it could be generalized as by equation (2.5), I is the Legendre-Fenchel transform of the pressure associated with the system.

6.3 The i.i.d. case

If Π has all its lines identical, then X is a sequence of i.i.d. random variables. Assume Π is a positive matrix, and let r denote the first line of Π: it is the law of those random variables. One should recover the Sanov theorem for i.i.d. random variables on a finite alphabet *i.e.* that L_n^X satisfies a LDP with rate function $H(\,\cdot\,|r)$.

Theorem 6.9 (Sanov)**.** *If (X_n) is a sequence of i.i.d. random variables of common law r over the finite state space S, then L_n^X satisfies a LDP with rate function $H(\cdot|r)$, where*

$$H(q|r) = \begin{cases} \sum_{i\in S} q_i \log \frac{q_i}{r_i} & \text{if } q \gg r \\ +\infty & \text{else.} \end{cases} \tag{6.4}$$

Proof. Assume r charges every state. Then **(Pos)** is satisfied. By Proposition 5.3, L_n^X satisfies a LDP with rate function $I = J = K = L$. Then,

$$\begin{aligned} J(q) &= \sup_{u>0} \sum_j q_j \log \frac{u_j}{(u\Pi)_j} \\ &= \sup_{u>0} \sum_j q_j \log \frac{u_j}{r_j |u|_1} \\ &= \sup_{\substack{u'>0 \\ |u'|=1}} \sum_j q_j \log \frac{u'_j}{r_j} \\ &= \sup_{u' \in \mathcal{M}_1(S)} \sum_j q_j \log \frac{u'_j}{r_j}. \end{aligned}$$

The supremum is reached at $u' = q$ because

$$\sum_j q_j \log \frac{q_j}{r_j} - \sum_j q_j \log \frac{u'_j}{r_j} = \sum_j q_j \log \frac{q_j}{u'_j} = H(q|u') \geq 0,$$

with equality if and only if $u' = q$. Thus

$$J(q) = \sum_j q_j \log \frac{q_j}{r_j} = H(q|r). \tag{6.5}$$

This achieves to prove the Sanov theorem for i.i.d random variables over a finite alphabet if r charges every state.

Whenever it does not, Π is not irreducible, but one can restrict the computation to a bloc of Π that is irreducible and positive, starting from time $n = 2$. □

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